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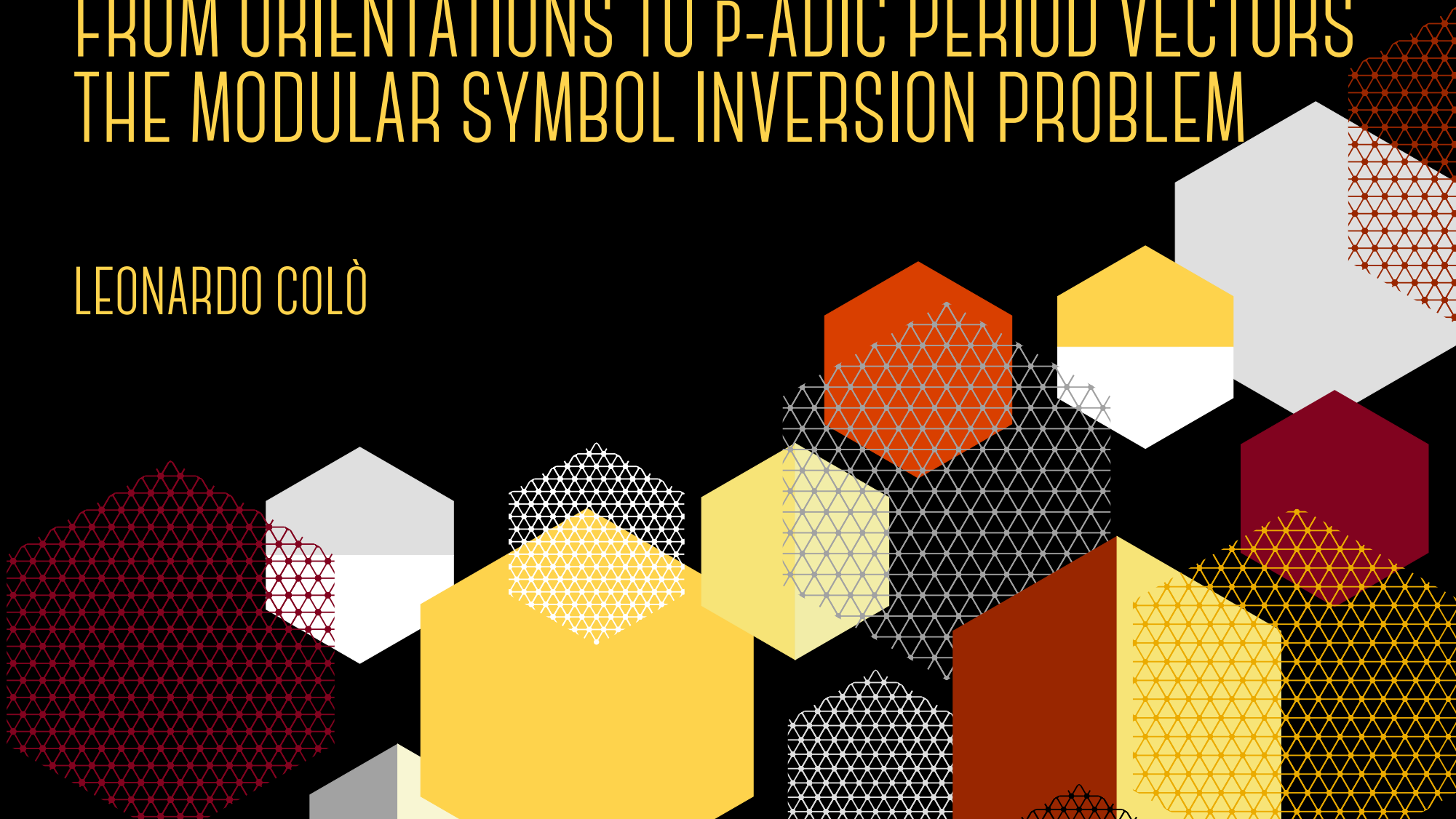
SEMINAIRE I2M

UNIVERSITY OF WATERLOO

FROM ORIENTATIONS TO p-ADIC PERIOD VECTORS THE MODULAR SYMBOL INVERSION PROBLEM

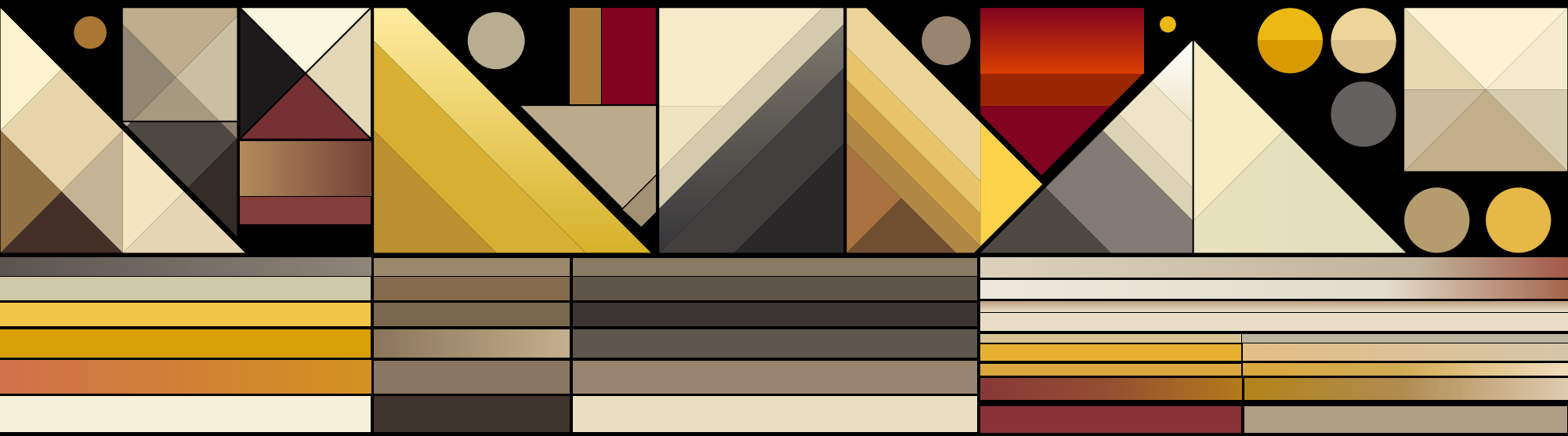
LEONARDO COLÒ



CONTENTS

- Supersingular isogeny graphs.
- Bruhat-Tits trees.
- Modular curves and level structures.
- Attaching modular symbols to orientations.
- p-adic integrals.
- Cryptographic constructions.

SUPERSINGULAR ELLIPTIC CURVES & ORIENTATIONS



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Elliptic curves

An elliptic curve over a field k (char $\neq 2, 3$):

$$E : y^2 = x^3 + Ax + B, \qquad \Delta = 4A^3 + 27B^2 \neq 0.$$

- Isogeny: non-constant algebraic group morphism $\varphi : E_1 \to E_2$.
- Kernel-theorem: if $\gcd(\deg \varphi, \mathrm{char}(k)) = 1$, then φ is uniquely determined by finite subgroup $\ker \varphi$.
- Dual isogeny and degree multiplication formula: $\widehat{\varphi} \circ \varphi = [\deg \varphi]$.

ISOGENY GRAPHS

Definition

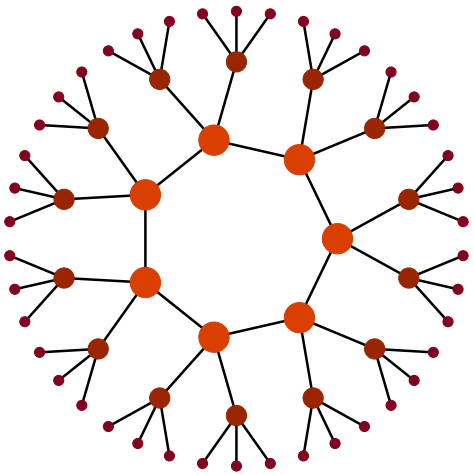
Given an elliptic curve E over k, and a finite set of primes S, we can associate an isogeny graph $G_S(E)$

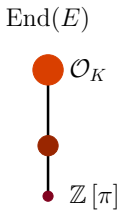
- whose vertices are elliptic curves isogenous to E over $\bar{k}$, and
- whose edges are isogenies of degree $\ell \in S$.

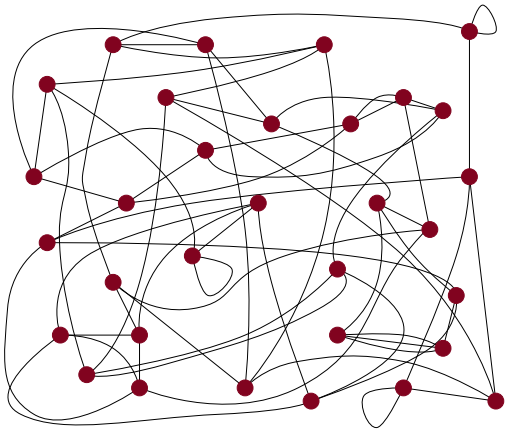
If $S = \{\ell\}$, then we write $G_\ell(E)$, the ℓ-isogeny graph.

The vertices are defined up to $\bar{k}$-isomorphism and the edges from a given vertex are defined up to a $\bar{k}$-isomorphism of the codomain.

The ℓ-isogeny graph of E is $(\ell + 1)$-regular (as a directed multigraph).







ORDERS AND ORIENTATIONS

Let E/k be a supersingular elliptic curve over $k = \mathbb{F}_{p^2}$, let K be an imaginary quadratic field with maximal order $\mathcal{O}_K$, and define

$$\mathrm{End}^0(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then $\mathrm{End}^0(E)$ is a quaternion algebra over $\mathbb{Q}$, ramified at p, which admits an embedding of K if and only if p is ramified or inert in K.

Definition

A *K-orientation* on an elliptic curve E/k is a homomorphism

$$\iota : K \hookrightarrow \mathrm{End}^0(E).$$

An *$\mathcal{O}$-orientation* on E is a K-orientation such that $\iota(\mathcal{O})$ is contained in $\mathrm{End}(E)$. An $\mathcal{O}$-orientation is *primitive* if

$$\iota : \mathcal{O} \to \iota(K) \cap \mathrm{End}(E)$$

is an isomorphism.

IDEAL CLASSES AND ORIENTED SUPERSINGULAR CURVES

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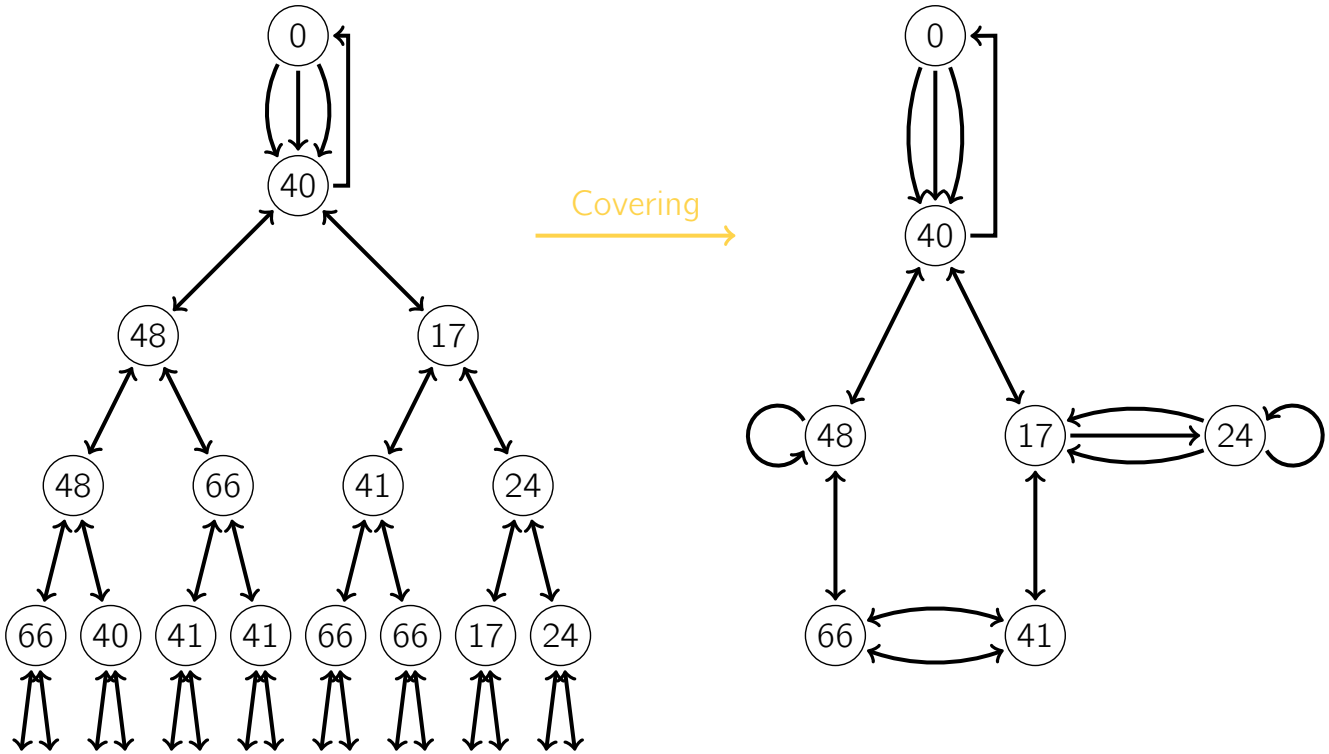
Theorem

The category of K-oriented supersingular elliptic curves (E, ι), whose morphisms are isogenies commuting with the K-orientations, is equivalent to the category of elliptic curves with CM by K.



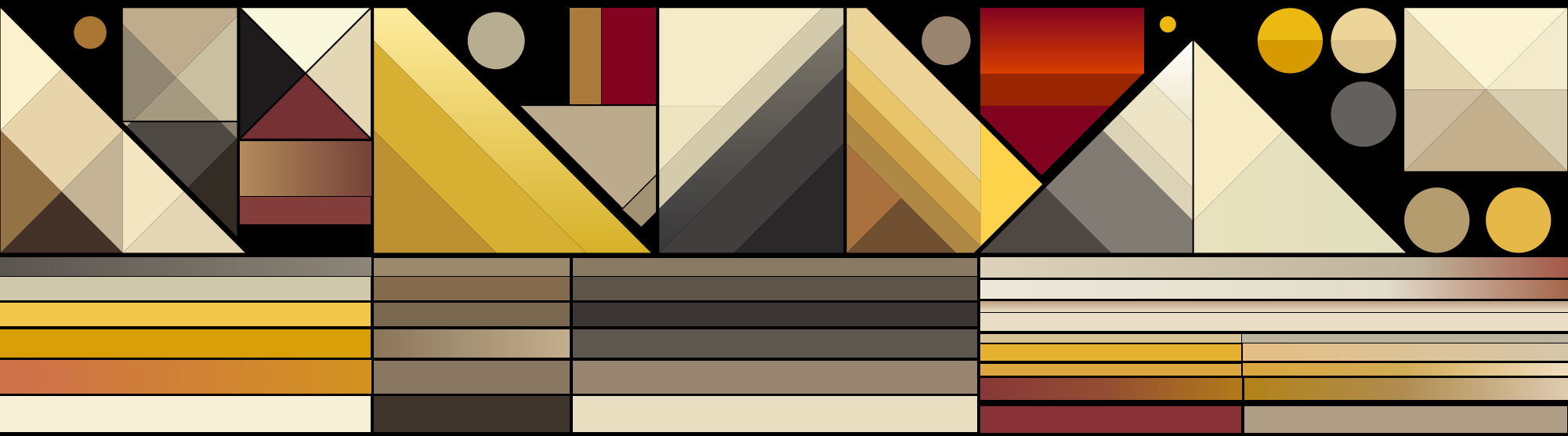
Definition

Let E be a supersingular elliptic curve, and let for a prime $\ell \neq p$ we define the ℓ-adic Tate module:

$$T_\ell(E) = \varprojlim E[\ell^n] \cong \mathbb{Z}_\ell^2.$$

- Every endomorphism of E acts $\mathbb{Z}_\ell$-linearly on $T_\ell(E)$.
- $\ell \neq p$: good reduction $\Rightarrow$ $T_\ell(E)$ free rank 2.
- Ideal classes act as elements of $\mathrm{GL}_2(\mathbb{Z}_\ell)$.
- An optimal embedding $\iota : \mathcal{O} \to \mathrm{End}(E)$ induces an action of $\mathcal{O}$ on $T_\ell(E)$. Thus the data of an oriented supersingular curve (E, ι) determines a $\mathcal{O}$-stable lattice in $V_\ell(E) = T_\ell \otimes \mathbb{Q}$, and horizontal isogenies preserve this lattice structure.

SUPERSINGULAR ISOGENY GRAPHS & BRUHAT-TITS TREE



THE BRUHAT-TITS TREE

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Definition

The Bruhat-Tits tree associated to $\mathrm{PGL}_2(\mathbb{Q}_\ell)$ is the graph $\mathcal{B}_\ell$ such that

- homothety classes of lattices of $\mathbb{Q}_\ell^2$.
- edges represent set of pairs of adjacent homothety classes.

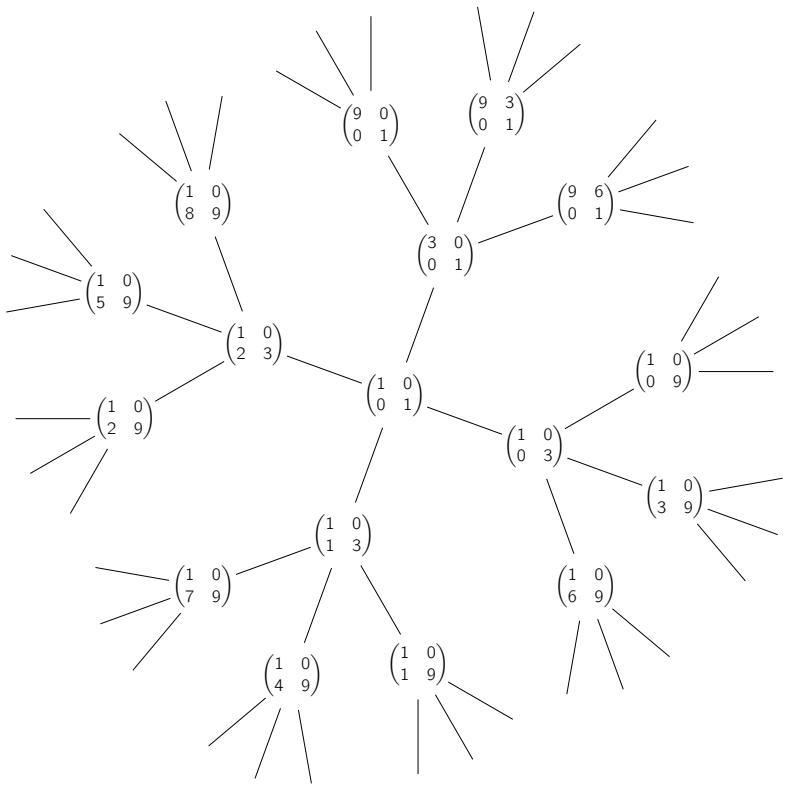
- A lattice L of $\mathbb{Q}_\ell^2$ is a free $\mathbb{Z}_\ell$-module of rank 2 in $\mathbb{Q}_\ell^2$
- We say that two lattices L_1 and L_2 are homothetic if there exists $\lambda \in \mathbb{Q}_\ell^\times$ such that $L_1 = \lambda L_2$.
- Two homothety classes $[L_1]$ and $[L_2]$ are adjacent if their representatives L_1 and L_2 can be chosen so that $\ell L_1 \subset L_2 \subset L_1$.

$\mathcal{B}_\ell$ is an Infinite $(\ell+1)$-regular tree encoding lattices in $\mathbb{Q}_\ell^2$.

THE BRUHAT-TITS TREE - EXAMPLE

There are several equivalent ways to define $\mathcal{B}_\ell$

- homothety classes of lattices of $\mathbb{Q}_\ell^2$.
- classes of matrices in $\mathrm{PGL}_2(\mathbb{Q}_\ell)/\mathrm{PGL}_2(\mathbb{Z}_\ell)$.
- maximal orders in the quaternion algebra $M_2(\mathbb{Q}_\ell)$.



ORIENTED VERTICES

- Let $\mathcal{B}_\ell$ be denote the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_\ell)$.
- Let $\Gamma = R$, where $R = \mathrm{End}(E_0)$ is a fixed maximal order. Then Γ acts on $\mathcal{B}_\ell$ without inversion, and the quotient $\Gamma\backslash\mathcal{B}_\ell$ is a finite graph, canonically identified with the $\overline{\mathbb{F}}_p$-isogeny graph of supersingular elliptic curves.
- Let (E, ι) be a supersingular elliptic curve equipped with an embedding

$$\iota : \mathcal{O} \hookrightarrow \mathrm{End}(E).$$

This induces an $\mathcal{O} \otimes \mathbb{Z}_\ell$-module structure on the ℓ-adic Tate module:

$$T_\ell(E) \cong \mathbb{Z}_\ell^2, \qquad \iota(a) \cdot v \in T_\ell(E) \text{ for all } a \in \mathcal{O}.$$

A vertex in $\mathcal{T}_\ell$ now carries **additional information**:

$$(E, \iota) \quad \longleftrightarrow \quad [T_\ell(E), \text{ with its full order structure } \mathcal{O} \otimes \mathbb{Z}_\ell].$$

No vertices or edges disappear — instead, the tree becomes a **decorated tree** where each vertex encodes the conductor profile.

IDEAL CLASSES AS ORIENTED PATHS

Let (E_0, ι_0) be a fixed oriented supersingular curve and let $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})$ be an ideal class, with $\mathfrak{a}$ coprime to $p\ell$.

- Fix a basis of the ℓ-adic Tate module $T_\ell(E_0) = \langle P, Q \rangle \cong \mathbb{Z}_\ell^2$.
- The induced map on Tate modules is an automorphism $T_\ell(\phi_\mathfrak{a}) : T_\ell(E_0) \to T_\ell(E_\mathfrak{a})$ represented by a matrix

$$M_\mathfrak{a} \in \mathrm{GL}_2(\mathbb{Z}_\ell)$$

  in the basis (P, Q).
- For each $n \geq 1$, the reduction of $M_\mathfrak{a}$ modulo ℓ^n describes the action on the finite module $E_0[\ell^n] \cong (\mathbb{Z}/\ell^n\mathbb{Z})^2$.

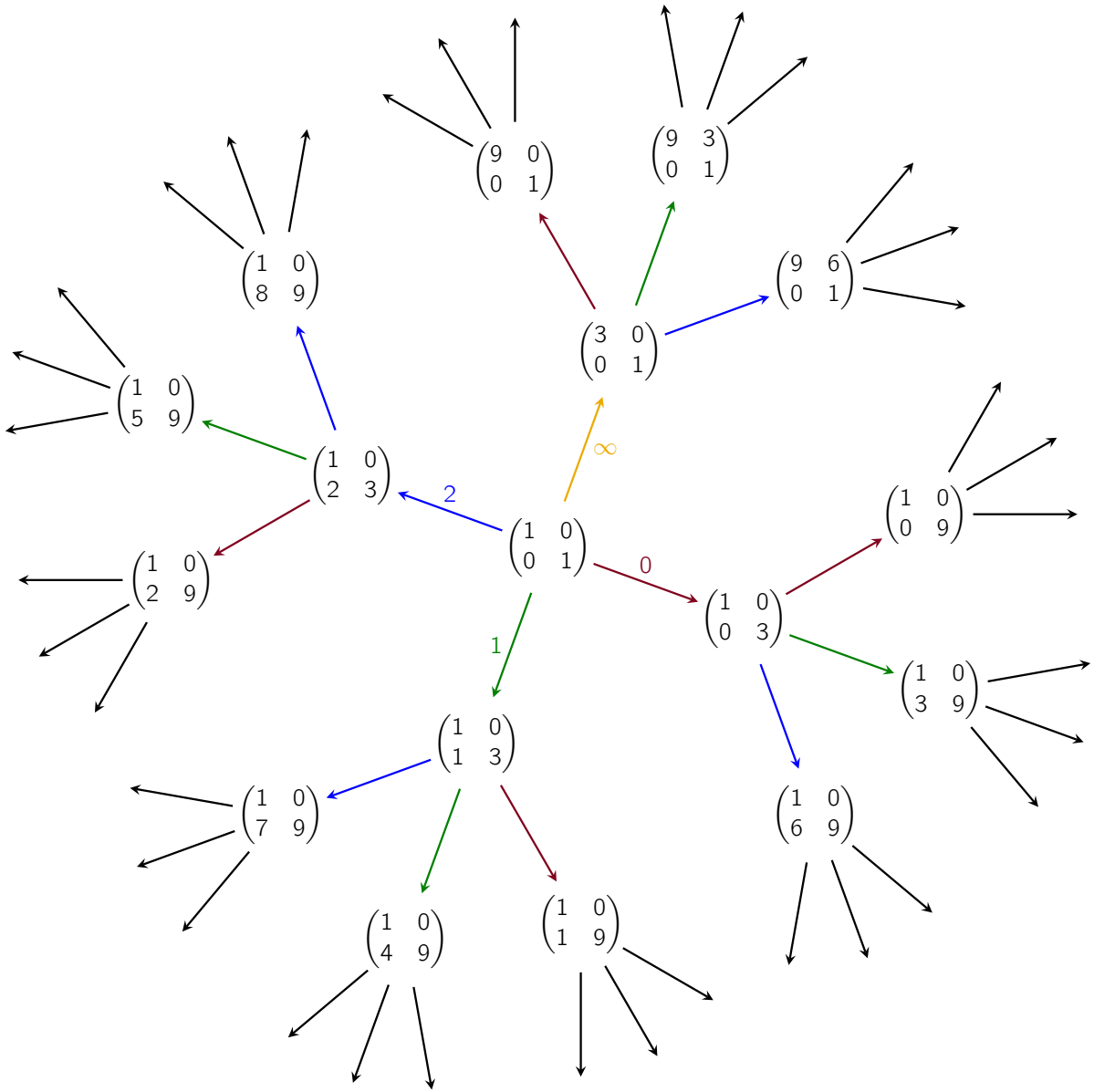
The sequence of digits

$$(c_0, c_1, c_2, \dots)$$

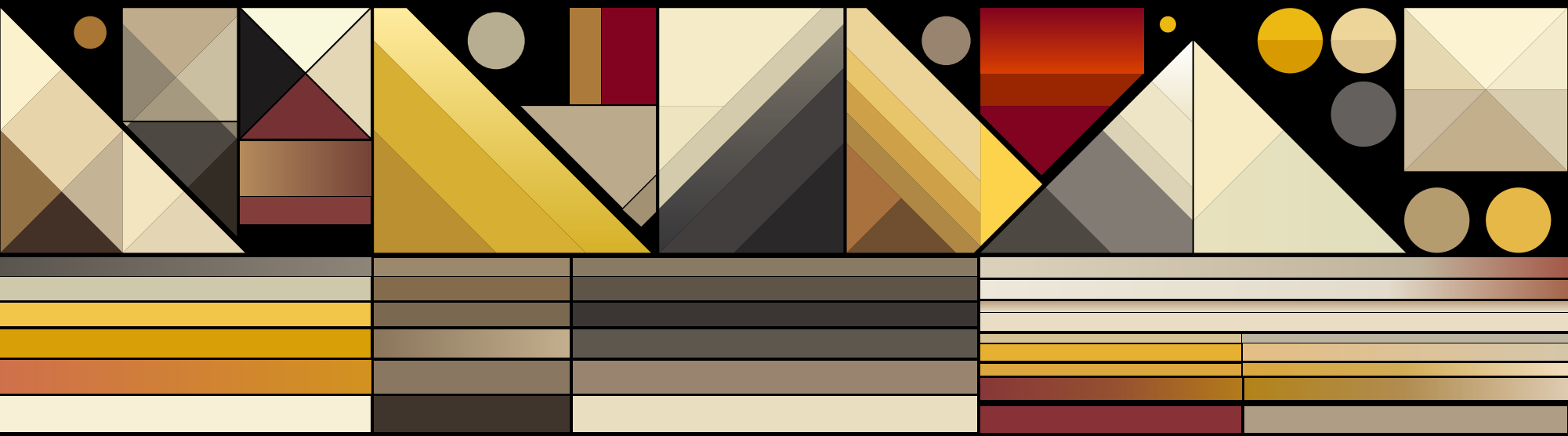
encodes a unique non-backtracking path in the Bruhat–Tits tree $\mathcal{T}_\ell$ starting at the vertex of (E_0, ι_0). Thus an ideal class $[\mathfrak{a}]$ determines a **canonical oriented path**

$$[\mathfrak{a}] \longleftrightarrow \text{ geodesic walk in } \mathcal{T}_\ell \text{ from } (E_0, \iota_0) \text{ to } (E_\mathfrak{a}, \iota_\mathfrak{a}).$$

NAVIGATING THE BRUHAT-TITS TREE - EXAMPLE



MODULAR SYMBOLS AND RELATIVE HOMOLOGY



MODULAR GROUP

The group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\}$$

acts by fractional linear Mobius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}$$

on the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0 \}$$

equipped with its standard complex analytic structure.

We define the modular group as the quotient group

$$\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$$

CONGRUENCE SUBGROUPS

We define the principal congruence subgroup as the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

A subgroup Γ of $\mathrm{SL}_2(N)$ is called a congruence subgroup if it contains $\Gamma(N)$ for some N. Some important examples of congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

MODULAR FORMS AND CUSP FORMS OF WEIGHT 2

A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a *weight-2 modular form* for $\Gamma_0(N)$ if

$$f(\gamma z)\,(cz+d)^{-2} = f(z) \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and f is holomorphic at the cusps.

The space of such forms is denoted $M_2(\Gamma_0(N))$.

A form $f \in M_2(\Gamma_0(N))$ is a *cusp form* if in its q-expansion at every cusp, the constant term vanishes. The subspace of cusp forms is denoted $S_2(\Gamma_0(N))$. To every $f \in S_2(\Gamma_0(N))$ we attach a differential

$$\omega_f = f(z)\,dz,$$

which is holomorphic on the modular curve $X_0(N)$.

HECKE OPERATORS ON MODULAR FORMS

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For each integer $n \geq 1$, the *Hecke operator* T_n acts on $M_2(\Gamma_0(N))$ and $S_2(\Gamma_0(N))$ via double coset correspondences. When $(n, N) = 1$, one has the formula

$$T_n(f)(z) = n \sum_{\substack{ad=n \\ 0 \leq b < d}} d^{-2} \, f\left(\frac{az+b}{d}\right).$$

These operators commute, are normal with respect to the Petersson inner product, and preserve $S_2(\Gamma_0(N))$. They play a central role in linking analytic and arithmetic constructions.

THE MODULAR CURVE $X_0(N)$ AND ITS CUSPS

Let $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be the congruence subgroup of level N defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

The modular curve $X_0(N)$ is the compact Riemann surface (and smooth projective algebraic curve over $\mathbb{Q}$) obtained by compactifying the quotient

$$Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$$

by adding finitely many cusps C, corresponding to the $\Gamma_0(N)$-orbits in $\mathbb{P}^1(\mathbb{Q})$. We denote by $g = g(X_0(N))$ the genus of $X_0(N)$ and by

$$C = \{c_1, \ldots, c_c\}$$

the finite set of cusps, where $c = \#C$.

RELATIVE HOMOLOGY & MANIN SYMBOLS

We let H denote the relative homology group

$$H := H_1(X_0(N), C; \mathbb{Z}).$$

Let Δ be the free abelian group on formal symbols $\{r \to s\}$ with $r, s \in \mathbb{P}^1(\mathbb{Q})$, modulo the relations

$$\{r \to s\} + \{s \to t\} + \{t \to r\} = 0 \qquad \text{for all } r, s, t \in \mathbb{P}^1(\mathbb{Q}),$$
$$\{r \to r\} = 0 \qquad \text{for all } r \in \mathbb{P}^1(\mathbb{Q}).$$

The group $\Gamma_0(N)$ acts on Δ by

$$\gamma \cdot \{r \to s\} := \{\gamma r \to \gamma s\}.$$

Proposition

There is a natural isomorphism of abelian groups

$$H_1(X_0(N), C; \mathbb{Z}) \cong \Delta_{\Gamma_0(N)},$$

where $\Delta_{\Gamma_0(N)} := \Delta / \langle \gamma \cdot x - x : \gamma \in \Gamma_0(N), x \in \Delta \rangle$

RANK FORMULA

The rank of the relative homology H can be expressed in terms of the genus and the number of cusps.

proposition

Let $g = g(X_0(N))$ be the genus of $X_0(N)$ and let $c = \#C$ be the number of cusps. Then

$$\operatorname{rank}_{\mathbb{Z}} H_1(X_0(N), C; \mathbb{Z}) = 2g + (c - 1).$$

Note that $\#C = \sum_{b|N} \varphi\left(\gcd\left(b, \frac{N}{b}\right)\right)$

HECKE OPERATORS ON MODULAR SYMBOLS

The Hecke algebra $\mathbb{T}$ of level $\Gamma_0(N)$ is generated by the usual Hecke operators T_ℓ for primes $\ell \nmid N$ and U_ℓ for $\ell \mid N$.

These operators act on cusp forms of weight 2 and level $\Gamma_0(N)$, but also on the homology H via correspondences on $X_0(N)$.

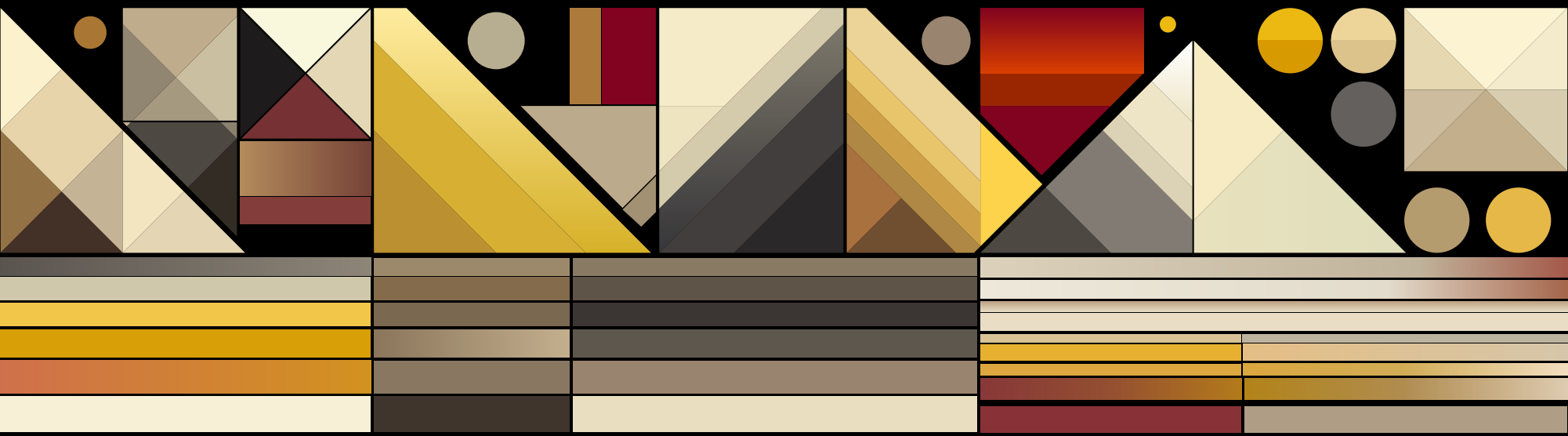
$$T_\ell \{\alpha \to \beta\} = \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \{\alpha \to \beta\} + \sum_{b=0}^{\ell-1} \begin{pmatrix} 1 & b \\ 0 & \ell \end{pmatrix} \{\alpha \to \beta\}$$

More precisely, each Hecke operator T_ℓ induces an endomorphism

$$T_\ell : H_1(X_0(N), C; \mathbb{Z}) \to H_1(X_0(N), C; \mathbb{Z}),$$

defined at the level of modular symbols by a finite linear combination of pullbacks and pushforwards along modular correspondences.

ATTACH MODULAR SYMBOL TO AN ORIENTATION



BRANDT HOMOLOGY

Input: An $\mathcal{O}$–oriented supersingular elliptic curve (E_0, ι_0) and an ideal class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})$.

- Supersingular curves with orientation correspond to left ideal classes in a definite quaternion algebra:

$$(E, \iota) \leftrightarrow [I] \in \mathcal{Cl}(\mathcal{O}_B).$$

- The Brandt module

$$\mathbb{B} = \mathbb{Z}[\mathcal{Cl}(\mathcal{O}_B)]$$

  carries a Hecke action.
- Via Jacquet–Langlands + Eichler–Shimura:

$$\iota_{\mathrm{JL}} : \mathbb{B} \hookrightarrow H_1(X_0(N), C; \mathbb{Z}) \otimes \mathbb{Q}.$$

- $\mathrm{Pic}(\mathcal{O})$ acts by permuting ideal classes; transporting a fixed base cycle gives:

$$\gamma^{(1)}([\mathfrak{a}]) := \rho([\mathfrak{a}])(\gamma_0) \in H_1(X_0(N), C; \mathbb{Z}).$$

GEOMETRIC GEODESIC CYCLES ON $X_0(N)$

Input: The same ideal class $[\mathfrak{a}]$.

- CM theory yields a Heegner point:

$$[\mathfrak{a}] \longmapsto x_{\mathfrak{a}} \in X_0(N)(\mathbb{C}),$$

compatible with the class group action.

- Fix:

$$x_0 := x_{\mathcal{O}_f}, \qquad c_\infty \in \text{cusps}.$$

Choose analytic paths:

$$\delta_{\mathfrak{a}} : x_{\mathfrak{a}} \to c_\infty, \qquad \delta_0 : x_0 \to c_\infty.$$

- Relative homology:

$$\gamma^{(2)}([\mathfrak{a}]) := \delta_{\mathfrak{a}} - \delta_0 \in H_1(X_0(N), C; \mathbb{Z}).$$

This is well-defined modulo absolute cycles and depends only on $[\mathfrak{a}]$.

BRUHAT-TITS GRAPH AND HARMONIC COCYCLES

Input: Same ideal class $[\mathfrak{a}]$.

- By Cerednik–Drinfeld uniformization, $X_0(N)$ admits a p-adic model whose skeleton is:

$$\Gamma \backslash \mathcal{T}_p,$$

  where $\mathcal{T}_p$ is the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$.
- Vertices correspond to oriented supersingular curves (or their p-adic lifts).
- The class-group action induces an *oriented path*:

$$v_0 \rightsquigarrow v_{\mathfrak{a}} \quad \text{in } \Gamma \backslash \mathcal{T}_p.$$

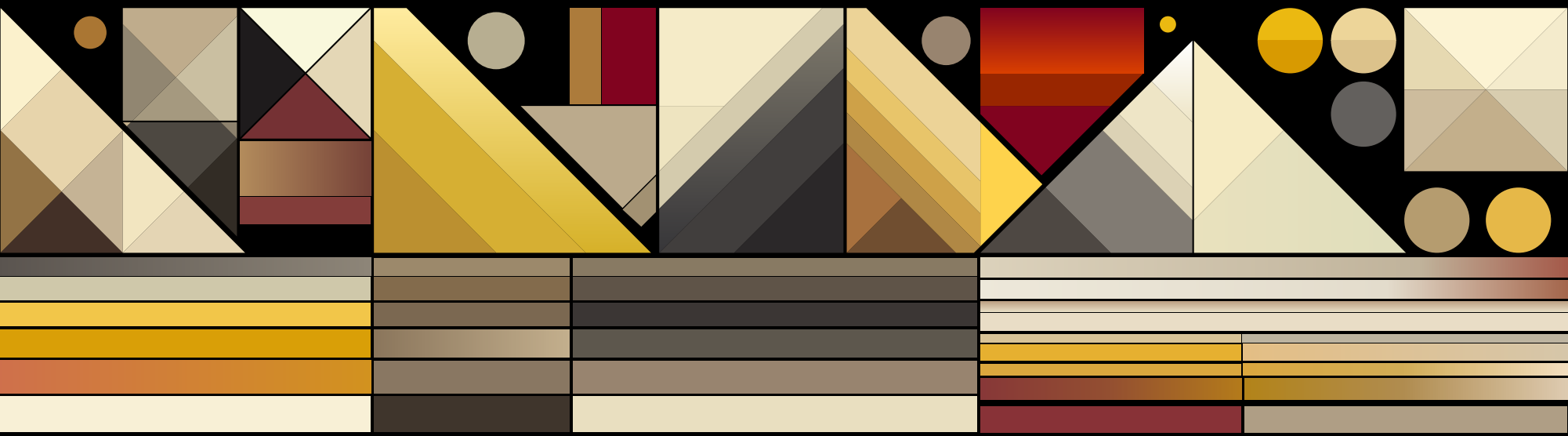
- Closing this path to a fixed base reference edge yields a graph cycle

$$c_{\mathfrak{a}} \in H_1(\Gamma \backslash \mathcal{T}_p; \mathbb{Z}).$$

- Via the harmonic cocycle isomorphism:

$$\gamma^{(3)}([\mathfrak{a}]) := \Phi(c_{\mathfrak{a}}) \in H_1(X_0(N), C; \mathbb{Z}).$$

p-ADIC PERIOD VECTORS AND COLEMAN INTEGRALS



WEIGHT-2 CUSP FORMS AND THE PERIOD PAIRING

$S_2(\Gamma_0(N))$ is the space of weight-2 cusp forms of level $\Gamma_0(N)$.

Definition

Let f be a weight-2 cusp form for $\Gamma_0(N)$. Classically, the *period pairing* between f and a homology class $\gamma \in H_1(X_0(N), C; \mathbb{Z})$ is the complex integral

$$\langle f, \gamma \rangle = \int_\gamma f(z)\, dz,$$

defined by integrating $f(z)\, dz$ along a singular 1-cycle representing γ in the upper half-plane model.

This extends linearly in both arguments and induces a perfect pairing between the cuspidal subspace of $H_1(X_0(N); \mathbb{Z})$ and the corresponding space of cusp forms.

DEFINING THE PERIOD VECTOR

Let $f_1, \ldots, f_d$ be a fixed collection of weight-2 cusp forms For $\gamma \in H$, we define the (infinite precision) p-adic period vector

$$\Pi(\gamma) := \left(\langle f_1, \gamma \rangle_p, \ldots, \langle f_d, \gamma \rangle_p\right) \in \mathbb{Q}_p^d.$$

Definition

Let q be a prime distinct from p and not dividing N. For $m \geq 1$ and $\gamma \in H$, the *truncated p-adic period vector* of γ is

$$\Pi_m(\gamma) := \left(\langle f_1, \gamma \rangle_p, \ldots, \langle f_d, \gamma \rangle_p\right) \bmod p^m \in (\mathbb{Z}/p^m\mathbb{Z})^d.$$

The map $\Pi_m : H \to (\mathbb{Z}/p^m\mathbb{Z})^d$ is $\mathbb{Z}$-linear, and its image is contained in a subgroup whose size depends on d, p, and m.

p-ADIC INTEGRATION - THE PROBLEM

Is there a Theory of p-adic integration?

We cannot use the same methods that we have in $\mathbb{R}$ or in $\mathbb{C}$ because of the totally disconnected topology of rigid analytic spaces.

Theorem (Coleman)

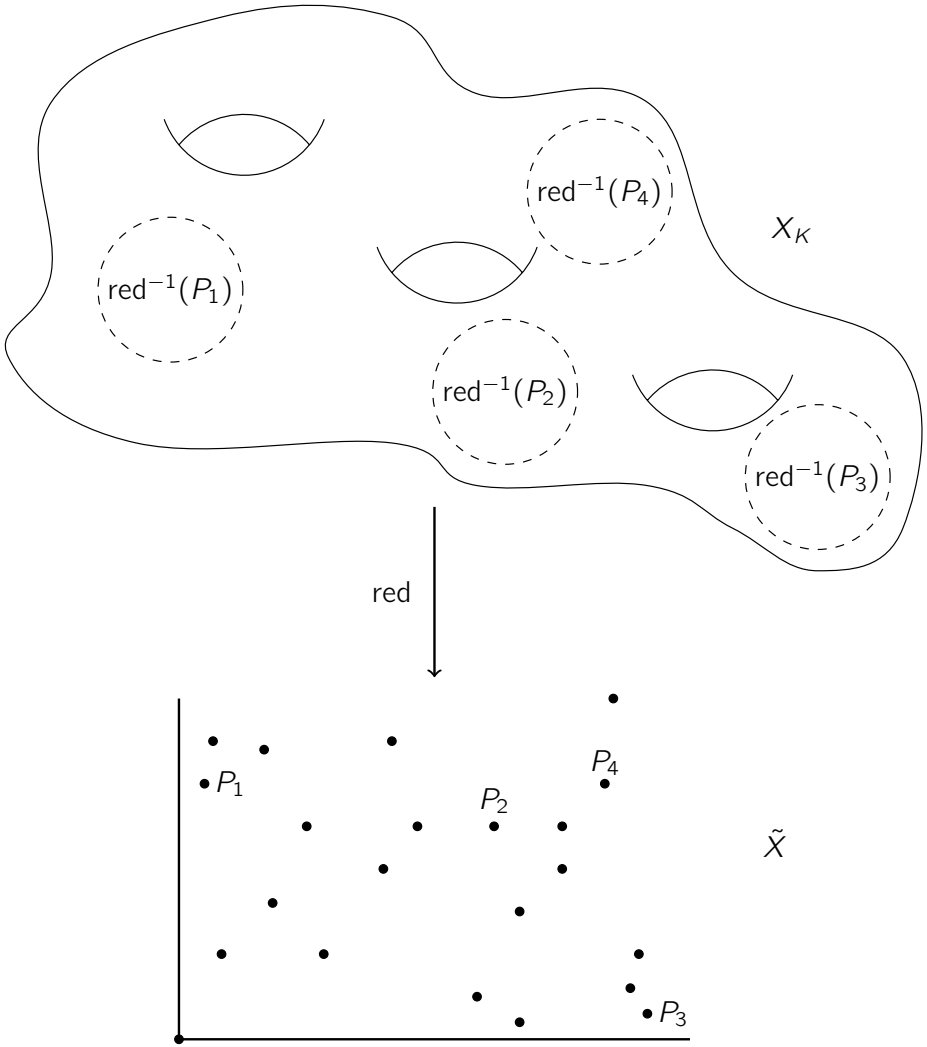
- Additivity at points: $\int_P^Q \omega + \int_Q^R \omega = \int_P^R \omega$
- Linearity on forms: $\lambda_1 \int_P^Q \omega_1 + \lambda_2 \int_P^Q \omega_2 = \int_P^Q (\lambda_1\omega_1 + \lambda_2\omega_2)$
- Change of variables: if X' is another rigid space and $\Psi : X \to X'$ is a rigid analytic map, then $\int_P^Q \Psi^*\omega' = \int_{\Psi(P)}^{\Psi(Q)} \omega'$.
- Fundamental Theorem of Calculus: $\int_P^Q df = f(Q) - f(P)$.

p-ADIC INTEGRATION - SETUP

- Let us consider a curve with good reduction.
- There is a natural reduction map

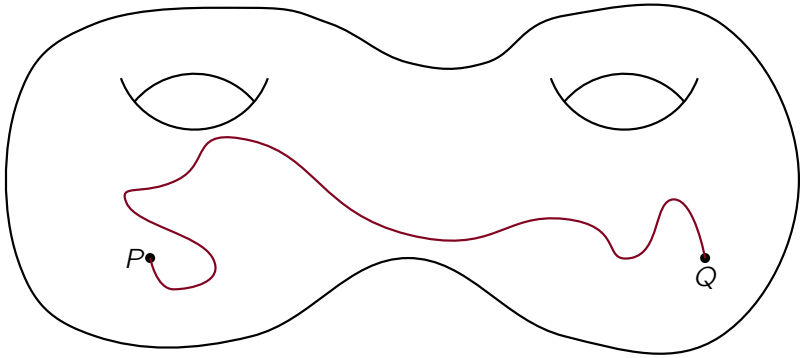
$$X \longrightarrow \tilde{X}(\mathbb{F})$$
$$x \longrightarrow \tilde{x} = x \cap A_0(X) \mod \mathfrak{p} A_0(X)$$

- The pre-image of any point of $\tilde{X}$ is a subspace of X_K isomorphic to an open unitary disk (residue disks).



COLEMAN INTEGRATION

There is no obvious way of integrating over affinoids.



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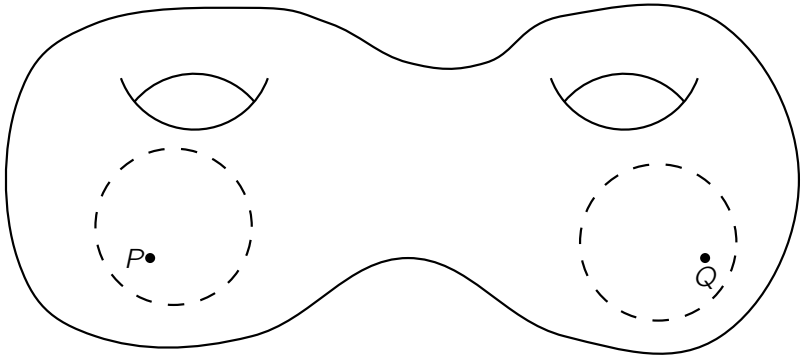
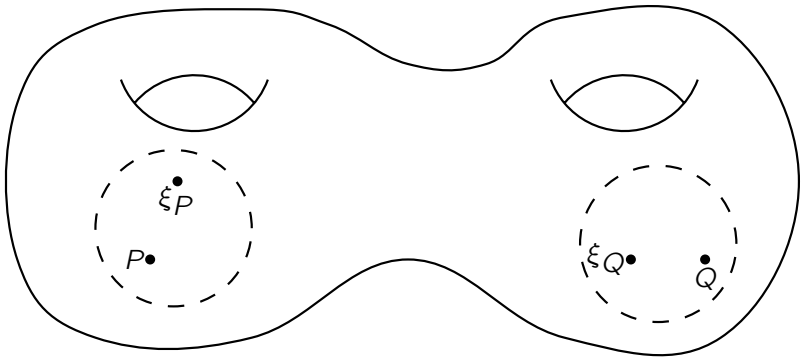


Coleman's Solution

- Cover the Affinoid space by residue disks.
- Integrate on each residue disk.
- **Problem:** Residue disks have no intersection.
- Connect integrals on different residue disks using Frobenius.

COLEMAN INTEGRATION

There is no obvious way of integrating over affinoids.



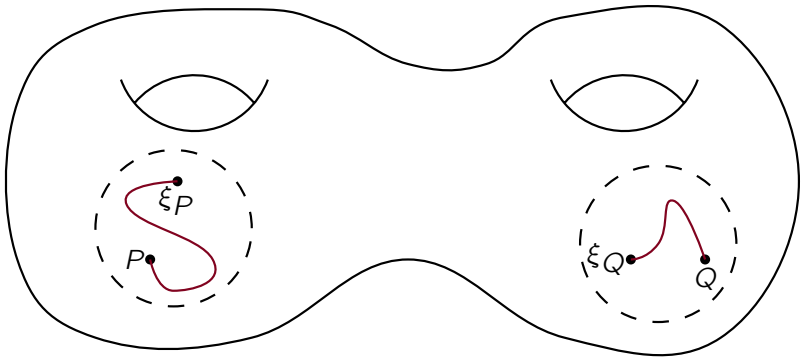
Coleman's Solution

- Find Teichmüller points

$$\int_P^Q \omega = \int_P^{\xi_P} \omega + \int_{\xi_P}^{\xi_Q} \omega + \int_{\xi_Q}^{Q} \omega$$

COLEMAN INTEGRATION

There is no obvious way of integrating over affinoids.



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- Find Teichmüller points
- Compute tiny integrals

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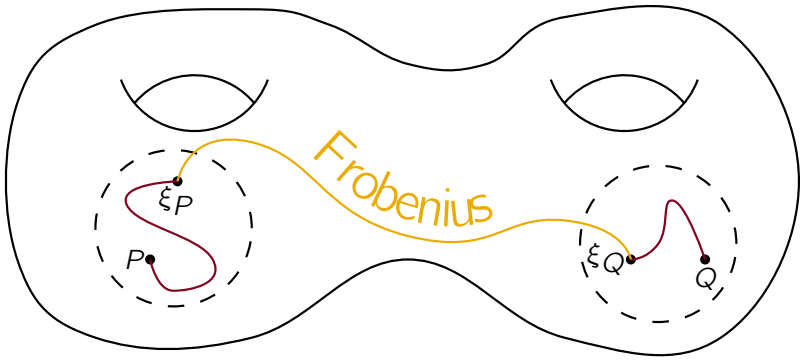

Coleman's Solution

- Find Teichmüller points
- Compute tiny integrals
- Connect integrals using Frobenius

$$\int_P^Q \omega = \int_P^{\xi_P} \omega + \int_{\xi_P}^{\xi_Q} \omega + \int_{\xi_Q}^{Q} \omega$$

COMPUTING INTEGRALS ON $X_\emptyset(N)$

L.COLÒ UWAT

Computing $\int_Q^R \omega$ for $Q, R \in X = X_0(N)$ and $\omega \in H^0(X, \Omega^1)$

We can leverage the existence of Hecke operators.

Algorithm (Chen, Kedlaya, Lau)

- Write $\int_Q^R \omega$ as a sum of tiny integrals.
- Find a basis of holomorphic 1-forms and a suitable uniformizer.
- Compute the action of Hecke operators on cusp forms and points
- Write 1-forms as a power series in the uniformizer. This involves algebraic approximation after solving a system of equations over $\mathbb{C}$.
- Formally integrate and evaluate at the end points.

BREAKING INTO TINY INTEGRALS

Let $X/\mathbb{Q}$ be a modular curve associated to a congruence subgroup Γ.

- The Hecke operator T_p, acts on the weight 2 cusp forms, which correspond to the holomorphic 1-forms:

$$T_p^* \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}$$

BREAKING INTO TINY INTEGRALS

Let $X/\mathbb{Q}$ be a modular curve associated to a congruence subgroup Γ.

- Since the Hecke operators are linear, integrating between the points Q and R gives:

$$\begin{pmatrix} \int_R^Q T_p^* \omega_1 \\ \vdots \\ \int_R^Q T_p^* \omega_g \end{pmatrix} = A \begin{pmatrix} \int_R^Q \omega_1 \\ \vdots \\ \int_R^Q \omega_g \end{pmatrix}$$

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- For any $\omega \in H^0(X, \Omega^1)$ we obtain

$$\int_Q^R T_p^*\omega = \int_{T_p^*(Q)}^{T_p^*(R)} \omega = \sum_{i=0}^{p} \int_{R_i}^{Q_i} \omega$$

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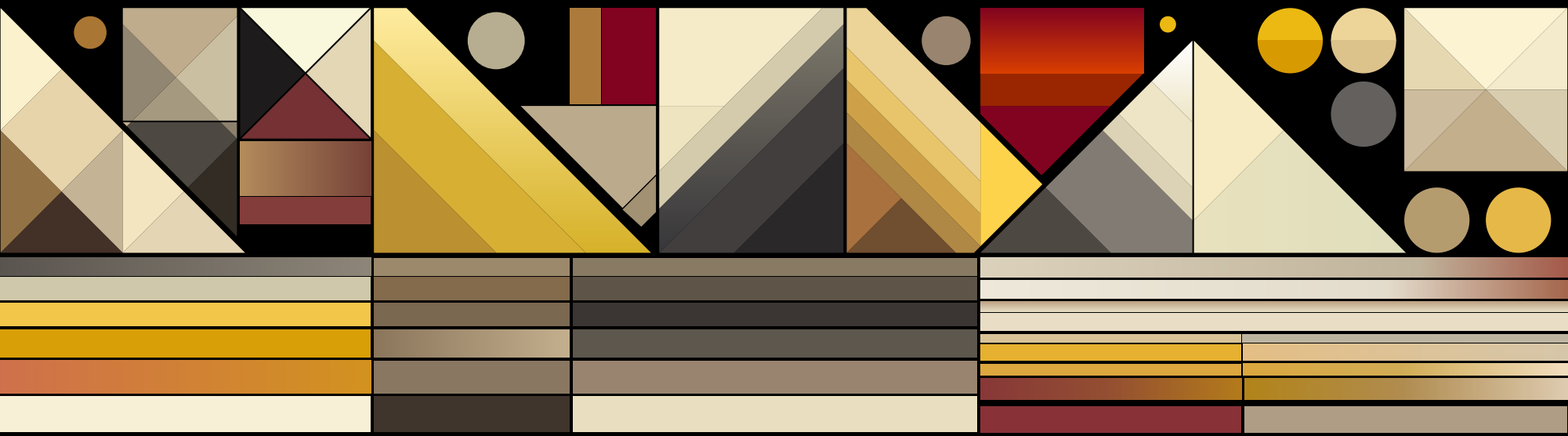
- We have the following fundamental equation

$$((p+1)I - A) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^p \int_{Q_i}^Q \omega_1 - \sum_{i=0}^p \int_{R_i}^R \omega_1 \\ \vdots \\ \sum_{i=0}^p \int_{Q_i}^Q \omega_g - \sum_{i=0}^p \int_{R_i}^R \omega_g \end{pmatrix}$$

THE MODULAR SYMBOL INVERSION PROBLEM MSI



PATH-ENCODED HOMOLOGY CLASSES

In practice, we work with homology classes arising from combinatorially simple "paths" in a suitable graph.

We assume that there is a distinguished finite generating set $\mathcal{S} = \{\sigma_1, \ldots, \sigma_r\}$ of H over $\mathbb{Z}$, where each σ_i corresponds to an "elementary step" in such a graph.

A *path of length* L is then an expression

$$\gamma = \sigma_{i_1} + \cdots + \sigma_{i_L},$$

subject to local constraints ensuring that successive steps fit together into a valid path (e.g. edges in the Bruhat–Tits tree match up at vertices). The set of all valid paths of length at most L is denoted $\mathcal{W}_L$.

DEFINITION OF MSI

Let $\Pi_m : H \to (\mathbb{Z}/p^m\mathbb{Z})^d$ be the truncated p-adic period map. Fix parameters L, p, m, d, and consider the following relation.

MSI relation

The *Modular Symbol Inversion relation* R_{MSI} is the subset of $(\mathbb{Z}/p^m\mathbb{Z})^d \times \mathcal{W}_L$ given by

$$R_{\mathrm{MSI}} := \{(y, \gamma) : \gamma \in \mathcal{W}_L,\ y = \Pi_m(\gamma)\}.$$

We will write $(y, \gamma) \in R_{\mathrm{MSI}}$ to mean that γ is a valid "short" homology preimage of y under Π_m.

MSI problem

Given a value $y \in (\mathbb{Z}/p^m\mathbb{Z})^d$ known (or promised) to satisfy $y = \Pi_m(\gamma^\star)$ for some unknown $\gamma^\star \in \mathcal{W}_L$, the *Modular Symbol Inversion (MSI) problem* is to find a $\gamma \in \mathcal{W}_L$ such that $(y, \gamma) \in R_{\mathrm{MSI}}$.

A FIAT-SHAMIR SIGNATURE BASED ON MSI

Public parameters:

$$(p, m, N, q), \quad \Pi_m : H_1(X_0(N), C; \mathbb{Z}) \to (\mathbb{Z}/p^m\mathbb{Z})^d.$$

Public key: $\mathbf{v} = \Pi_m(\gamma)$.
Secret key: short representative γ.

- **Commitment:** Pick random short $r \in H_1(X_0(N), C; \mathbb{Z})$, send
$$c = \Pi_m(r + \gamma).$$
- **Challenge:** Verifier samples
$$b \xleftarrow{\$} \{0, 1, \ldots, q-1\}.$$
- **Response:** Send
$$z = r + b\gamma.$$
Apply rejection sampling to ensure $\|z\|$ stays within bounds.

Verification:

$$\Pi_m(z) \stackrel{?}{\equiv} c - b \cdot \mathbf{v} \quad \text{in } (\mathbb{Z}/p^m\mathbb{Z})^d.$$

(Security relies on hardness of MSI: recovering γ from $v = \Pi_m(\gamma)$.)

NAIVE KEY EXCHANGE

Setup (public):

- Same global parameters (N, p, H', Π_m) as before.
- Same base class γ_0.

Alice picks: a secret class $\gamma_A \in H'$ and publishes:

$$v_A = \Pi_m(\gamma_A).$$

Bob picks: a secret class $\gamma_B \in H'$ and publishes:

$$v_B = \Pi_m(\gamma_B).$$

Shared key (intended):

$$K := \Pi_m(\gamma_A + \gamma_B),$$

computed as:

$$\boxed{K = v_A + v_B \pmod{p^m}},$$

using p-adic linearity of Π_m.

(Useful as a pedagogical toy model; not a secure KEX without masking, hashing, or nonlinearity.)

THANK YOU FOR THE ATTENTION