ORIENTATIONS & CLASS GROUP ACTIONS

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Crypto Research Seminar

CONTENTS

ISOGENY GRAPHS

Definition

Given an elliptic curve E over k, and a finite set of primes S, we can associate an isogeny graph $G_S(E)$

- whose vertices are elliptic curves isogenous to E over $\bar{k}$, and
- whose edges are isogenies of degree $\ell \in S$.

If $S = \{\ell\}$, then we write $G_\ell(E)$, the ℓ-isogeny graph.

The vertices are defined up to $\bar{k}$-isomorphism and the edges from a given vertex are defined up to a $\bar{k}$-isomorphism of the codomain.

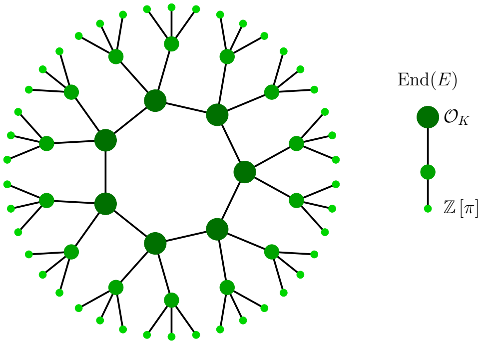
The ℓ-isogeny graph of E is $(\ell + 1)$-regular (as a directed multigraph).

ORDINARY ISOGENY GRAPHS: VOLCANOES

Let $\mathrm{End}(E) = \mathcal{O} \subseteq K$, an imaginary quadratic field. The class group $\mathrm{Cl}(\mathcal{O})$ acts faithfully and transitively on the set of elliptic curves with endomorphism ring $\mathcal{O}$:

$$E \longrightarrow E/E[\mathfrak{a}] \qquad E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \ \forall \alpha \in \mathfrak{a}\}$$

Thus, the CM isogeny graphs can be modelled by an equivalent category of fractional ideals of K.

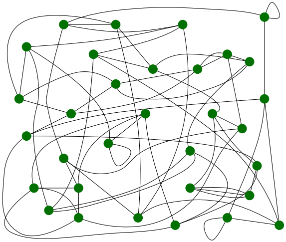


SUPERSINGULAR ISOGENY GRAPHS

The supersingular isogeny graphs are remarkable because the vertex sets are finite : there are $(p+1)/12+\epsilon_p$ curves. Moreover

- every supersingular elliptic curve can be defined over $\mathbb{F}_{p^2}$;
- all ℓ-isogenies are defined over $\mathbb{F}_{p^2}$;
- every endomorphism of E is defined over $\mathbb{F}_{p^2}$.

The lack of a commutative group acting on the set of supersingular elliptic curves/$\mathbb{F}_{p^2}$ makes the isogeny graph more complicated.



ORDINARY PROTOCOL - COUVEIGNES & ROSTOVTSEV-STOLBUNOV, 2006

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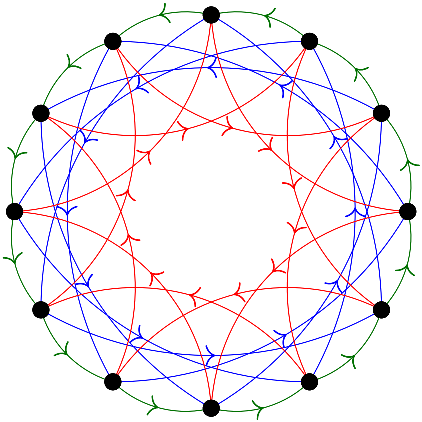

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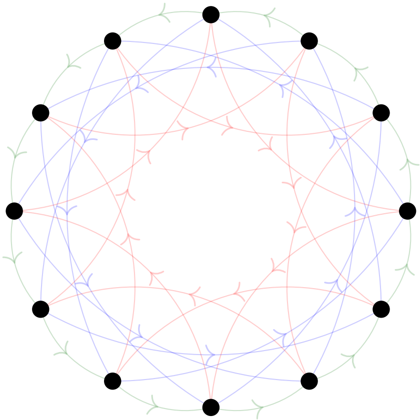

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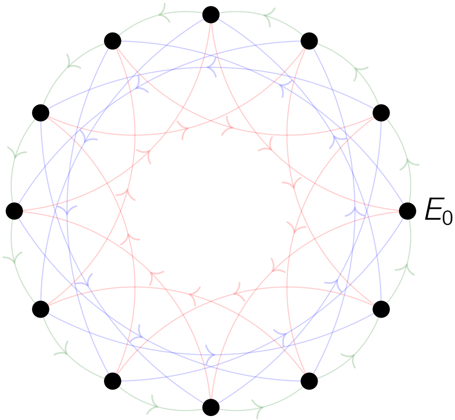

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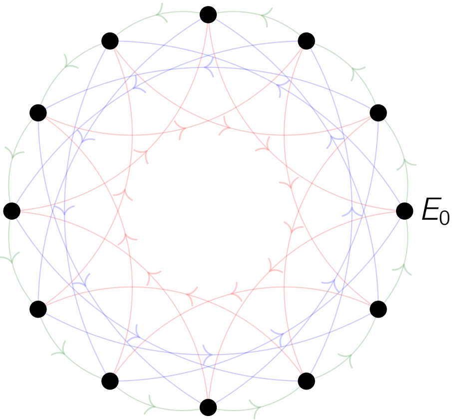

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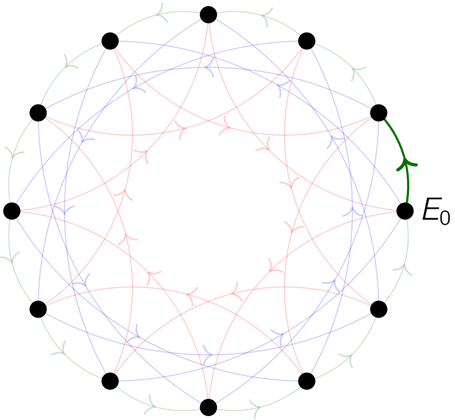

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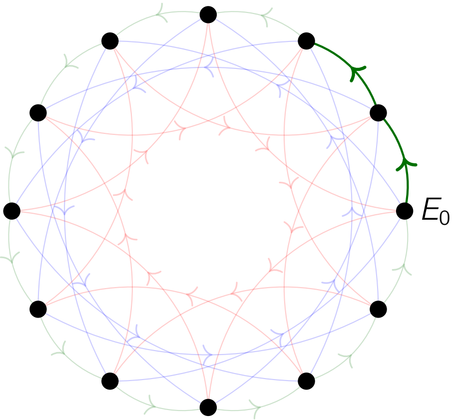

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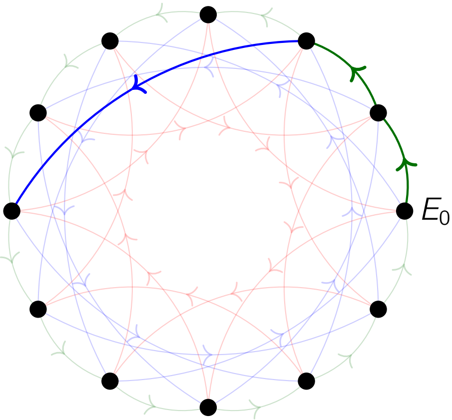

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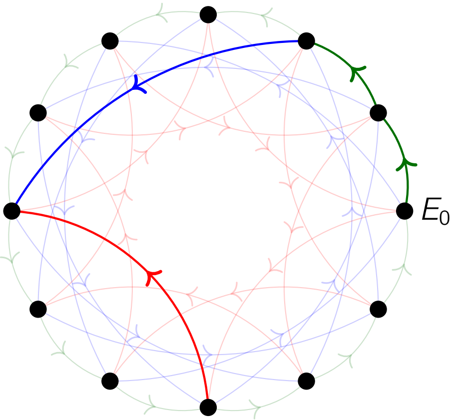

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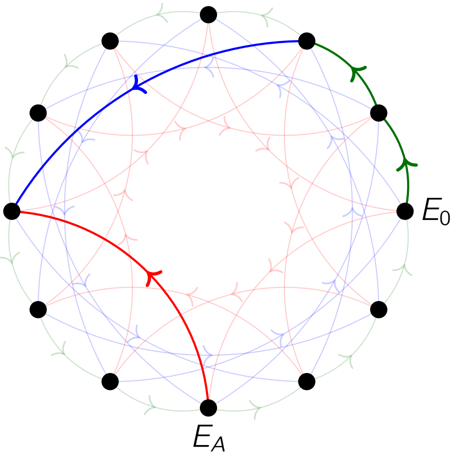

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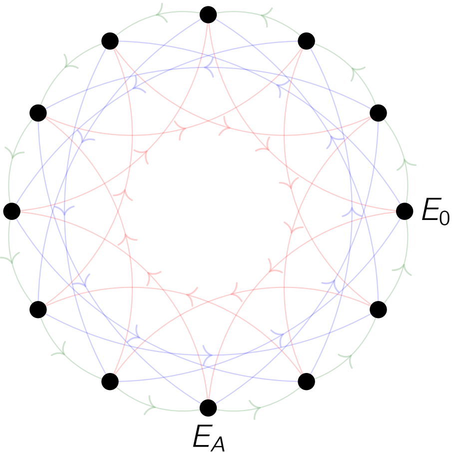

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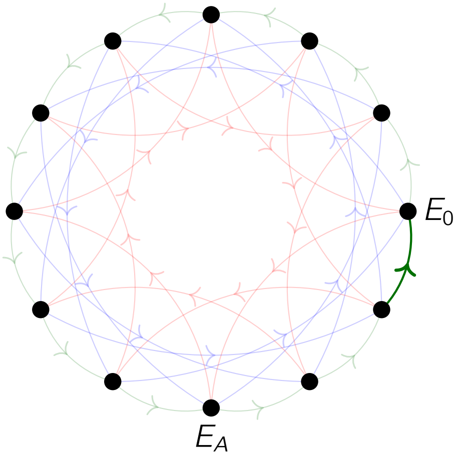

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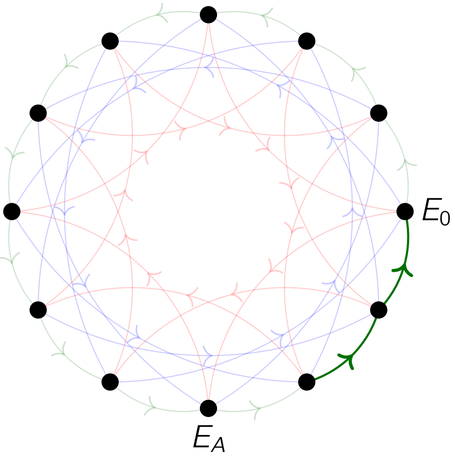

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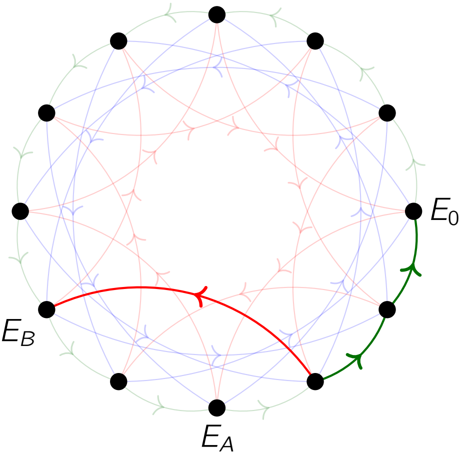

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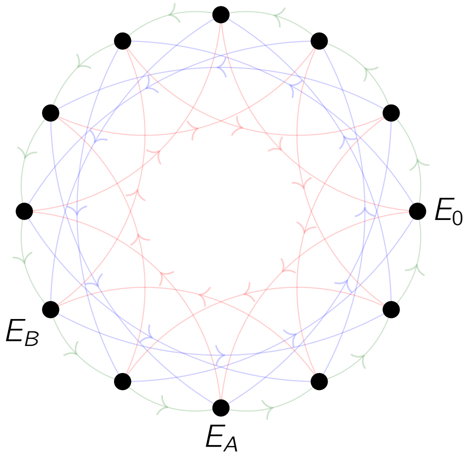

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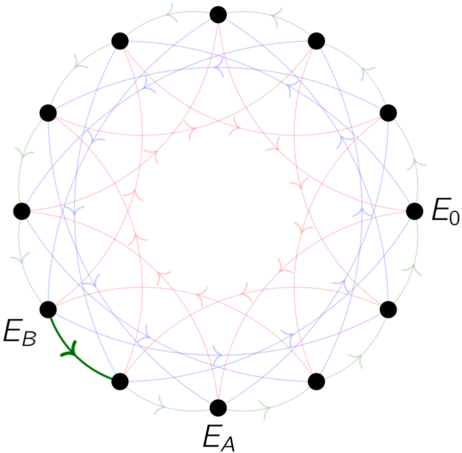

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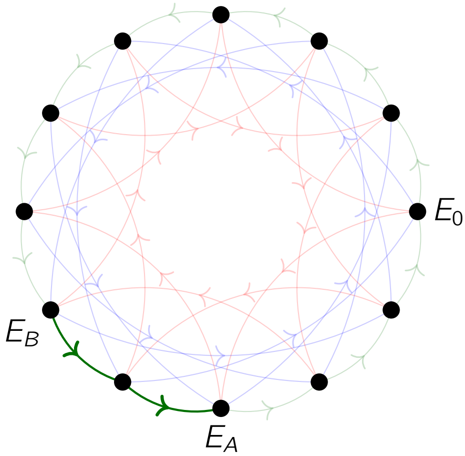

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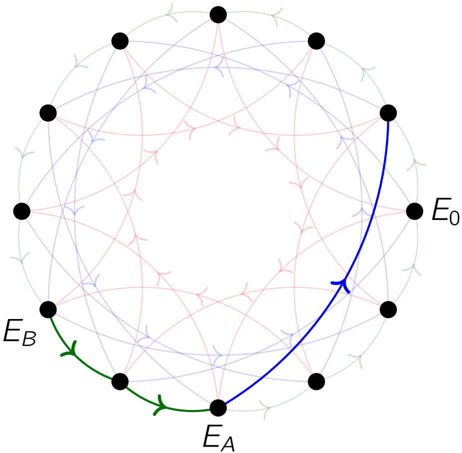

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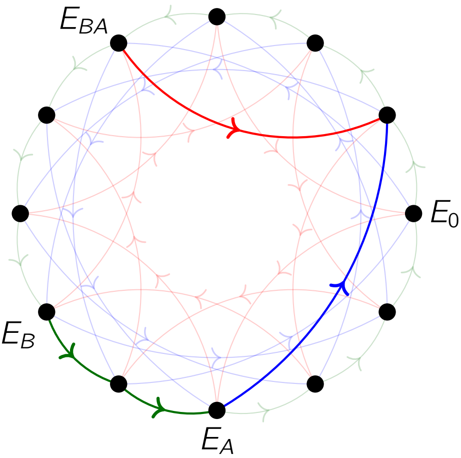

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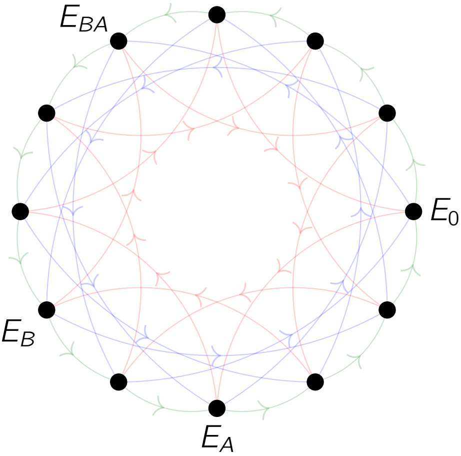

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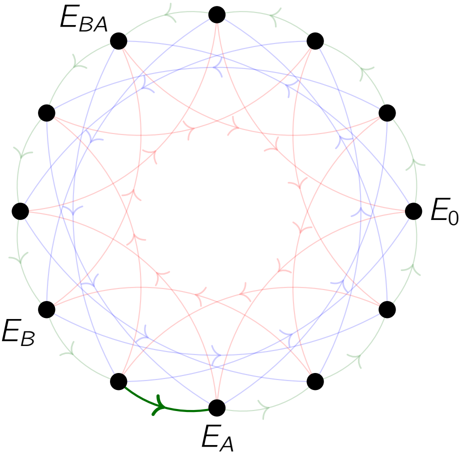

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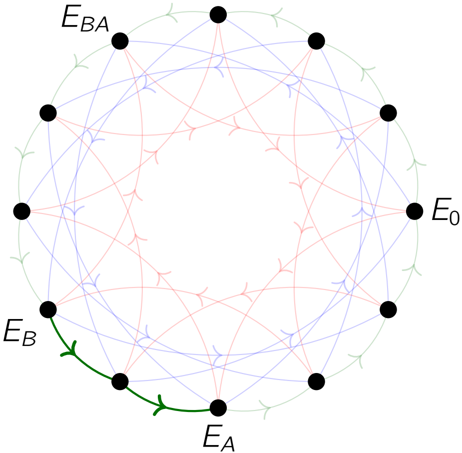

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L.COLÒ UWAT

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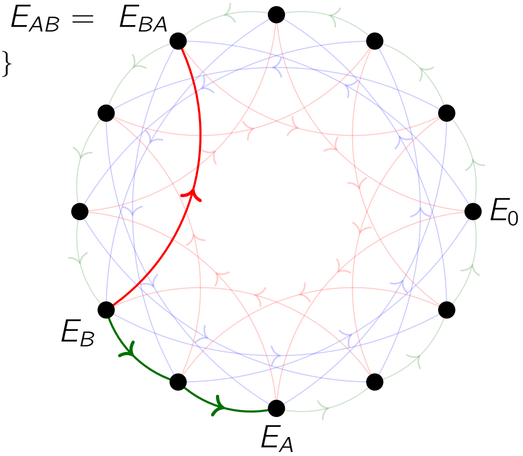

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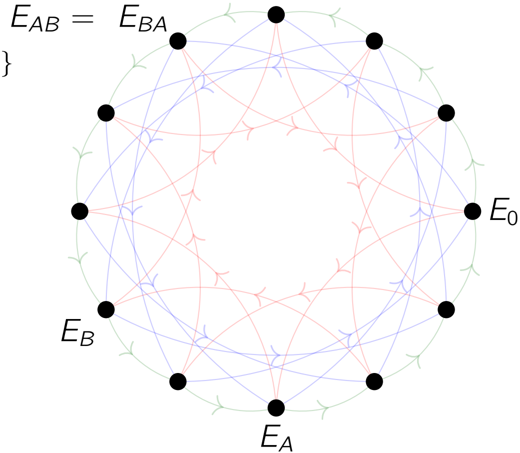


Alice

$\rho_A = (2, 1, -1)$

$\mathfrak{a} = \mathfrak{l}_1^2 \mathfrak{l}_2^1 \mathfrak{l}_3^{-1}$

Bob

$\rho_B = (-2, 0, 1)$

$\mathfrak{a} = \mathfrak{l}_1^{-2} \mathfrak{l}_3^1$

ORIENTATIONS & OSIDH

ORIENTATIONS

Let $\mathcal{O}$ be an order in an imaginary quadratic field K.
An $\mathcal{O}$-*orientation* on a supersingular elliptic curve E is an embedding

$$\iota : \mathcal{O} \hookrightarrow \mathrm{End}(E).$$

A K-*orientation* is an embedding

$$\iota : K \hookrightarrow \mathrm{End}^0(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

An $\mathcal{O}$-orientation is *primitive* if

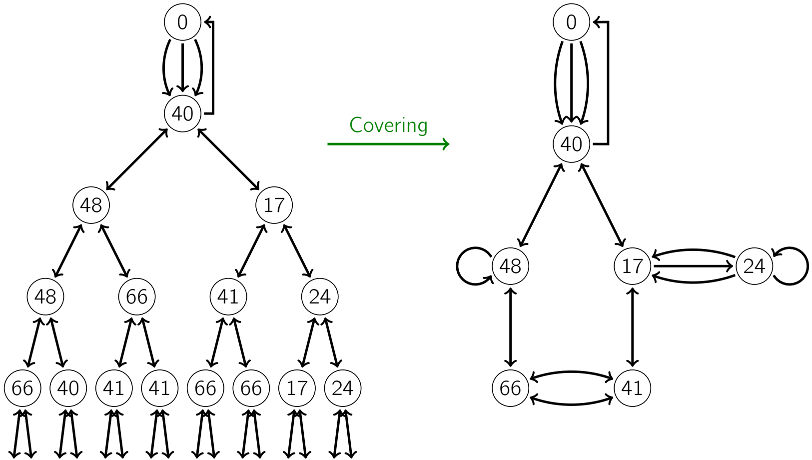
$$\mathcal{O} \simeq \mathrm{End}(E) \cap \iota(K).$$

Theorem

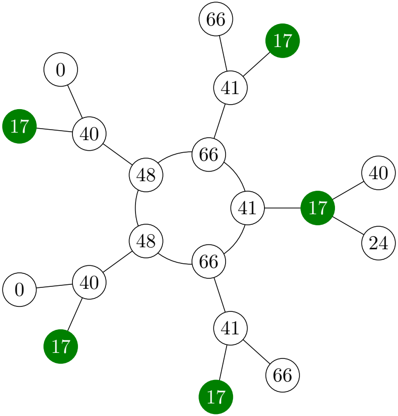
The category of K-oriented supersingular elliptic curves (E, ι), whose morphisms are isogenies commuting with the K-orientations, is equivalent to the category of elliptic curves with CM by K.

ORIENTED ISOGENY GRAPHS - AN EXAMPLE

Let $p = 71$ and $E_0/\mathbb{F}_{71}$ be the supersingular elliptic curve with $j(E) = 0$ oriented by the $\mathcal{O}_K = \mathbb{Z}[\omega]$, where $\omega^2 + \omega + 1 = 0$.
The orientation by $K = \mathbb{Q}[\omega]$ differentiates vertices in the descending paths from E_0, determining an infinite graph shown here to depth 4:



We let again $p = 71$ and we consider the isogeny graph oriented by $\mathbb{Z}[\omega_{79}]$ where ω_{79} generates the ring of integers of $\mathbb{Q}(\sqrt{-79})$.



CLASS GROUP ACTION

The set $\mathrm{SS}_{\mathcal{O}}(\rho)$ admits a transitive group action:

$$\begin{aligned} \mathcal{C}\ell(\mathcal{O}) \times \mathrm{SS}_{\mathcal{O}}(\rho) &\longrightarrow \mathrm{SS}_{\mathcal{O}}(\rho) \\ ([\mathfrak{a}], E) &\longmapsto [\mathfrak{a}] \cdot E = E/E[\mathfrak{a}] \end{aligned}$$

Proposition

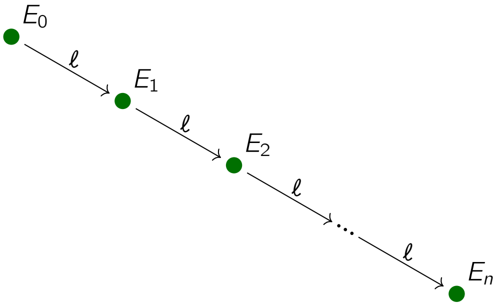
The set $\mathrm{SS}_{\mathcal{O}}^{pr}(\rho)$ is a torsor for the class group $\mathcal{C}\ell(\mathcal{O})$.

For fixed primitive ρ-oriented supersingular curve E, we get bijection of sets:

$$\mathcal{C}\ell(\mathcal{O}) \longrightarrow \mathrm{SS}_{\mathcal{O}}^{pr}(\rho)$$

EFFECTIVE CLASS GROUP ACTIONS

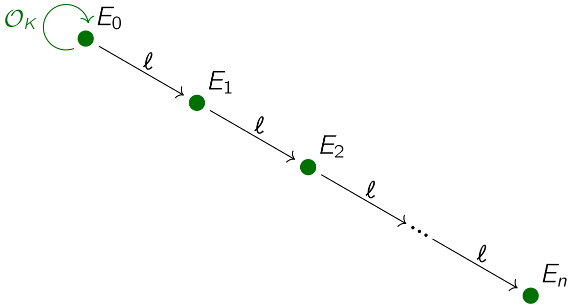
We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.



EFFECTIVE CLASS GROUP ACTIONS

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

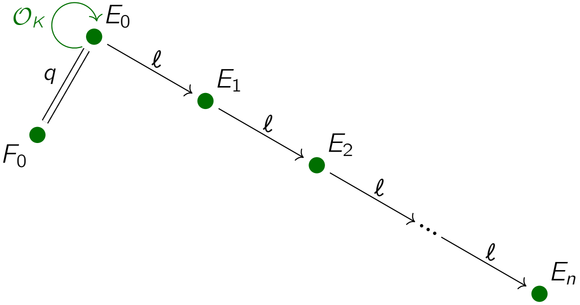
- For $\ell = 2$ (or 3) a suitable candidate for $\mathcal{O}_K$ could be the Gaussian integers $\mathbb{Z}[i]$ or the Eisenstein integers $\mathbb{Z}[\omega]$.



EFFECTIVE CLASS GROUP ACTIONS

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

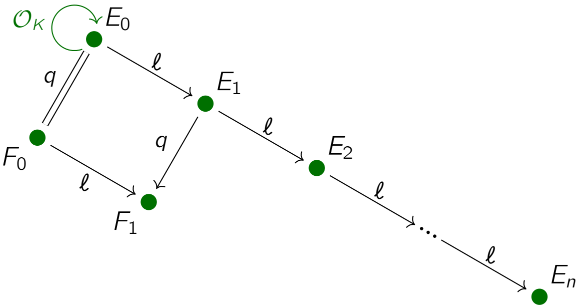
- Horizontal isogenies must be endomorphisms



Effective Class Group Actions

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

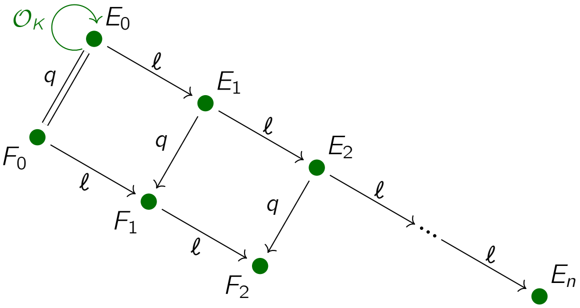
- We push forward our q-orientation obtaining F_1.



EFFECTIVE CLASS GROUP ACTIONS

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

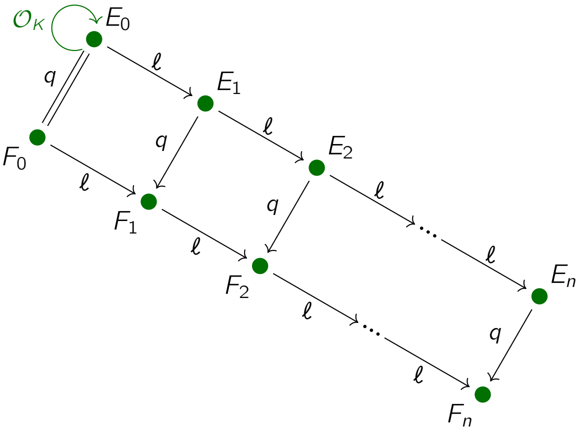
- We repeat the process for F_2.



EFFECTIVE CLASS GROUP ACTIONS

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

- And again till F_n.



OSIDH PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \subseteq \mathcal{O} \subseteq \mathrm{End}(E_n) \cap K \subseteq \mathcal{O}_K$

ALICE	BOB

OSIDH PROTOCOL

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	ALICE	**BOB**
Choose integers in a bound $[-r, r]$	$(e_1, \ldots, e_t)$	$(d_1, \ldots, d_t)$

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \subseteq \mathcal{O} \subseteq \mathrm{End}(E_n) \cap K \subseteq \mathcal{O}_K$

	ALICE	**BOB**
Choose integers in a bound $[-r, r]$	$(e_1, \ldots, e_t)$	$(d_1, \ldots, d_t)$
Construct an isogenous curve	$F_n = E_n/E_n[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}]$	$G_n = E_n/E_n[\mathfrak{p}_1^{d_1} \cdots \mathfrak{p}_t^{d_t}]$

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Construct an isogenous curve	$F_n = E_n / E_n\left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]$	$G_n = E_n / E_n\left[\mathfrak{p}_1^{d_1} \cdots \mathfrak{p}_t^{d_t}\right]$
Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$

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... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \ldots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \ldots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$

OSIDH PROTOCOL

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Exchange data	G_n+directions	F_n+directions

OSIDH PROTOCOL

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \ldots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \ldots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions
Compute shared data	Takes e_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.	Takes d_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.

OSIDH PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \subseteq \mathcal{O} \subseteq \mathrm{End}(E_n) \cap K \subseteq \mathcal{O}_K$

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \ldots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \ldots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions
Compute shared data	Takes e_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.	Takes d_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.

In the end, they share $H_n = E_n / E_n \left[\mathfrak{p}_1^{e_1+d_1} \cdot \ldots \cdot \mathfrak{p}_t^{e_t+d_t}\right]$

OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

$p = 10007$

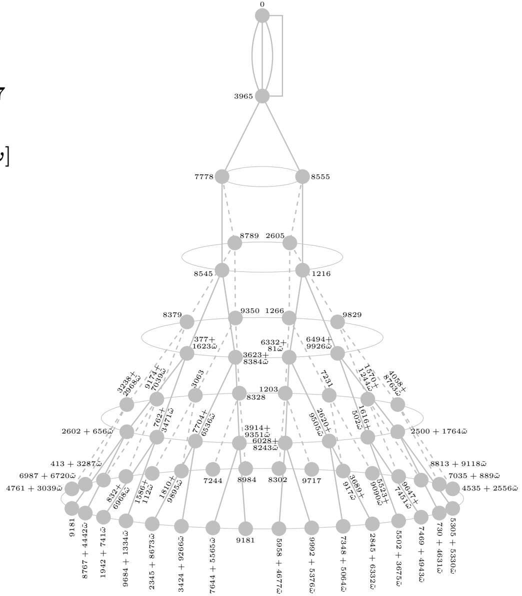
$\ell = 2$

$\mathcal{O}_K = \mathbb{Z}[\omega]$

$\ell_1 = 13$

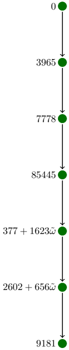
$\ell_2 = 31$

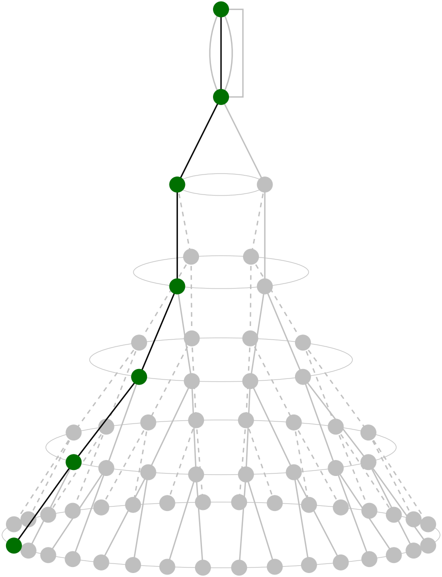
$\ell_3 = 43$



OSIDH PROTOCOL - AN EXAMPLE

Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$





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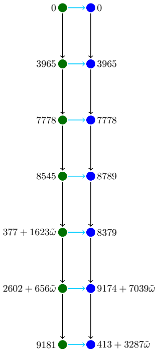
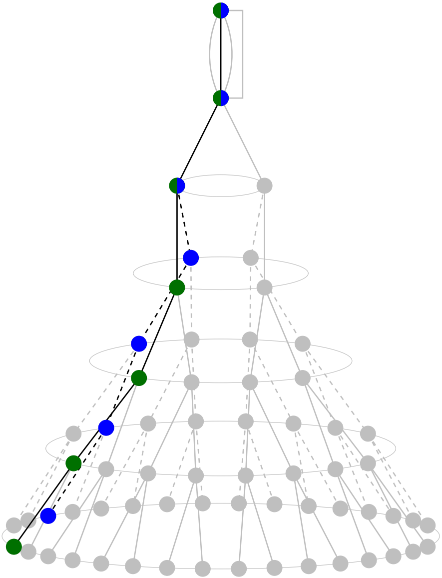
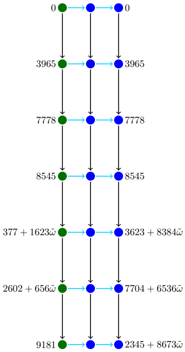


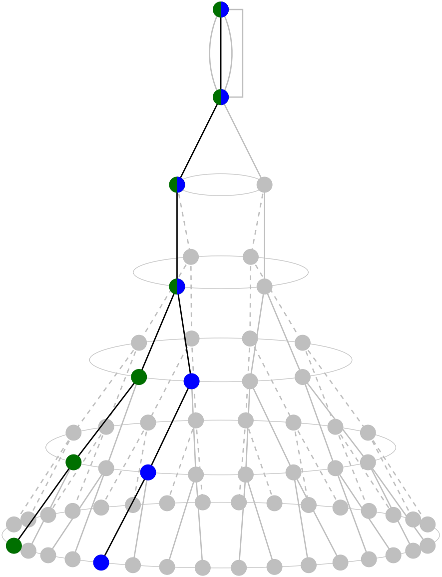


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$





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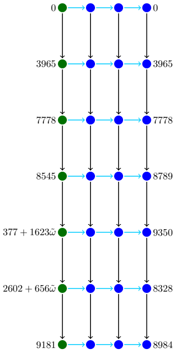

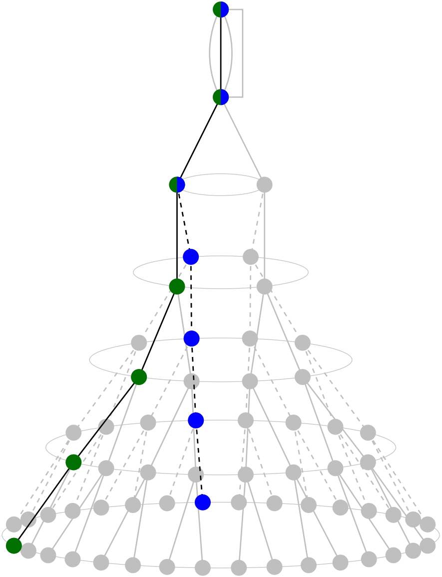

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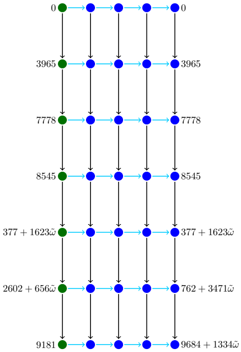

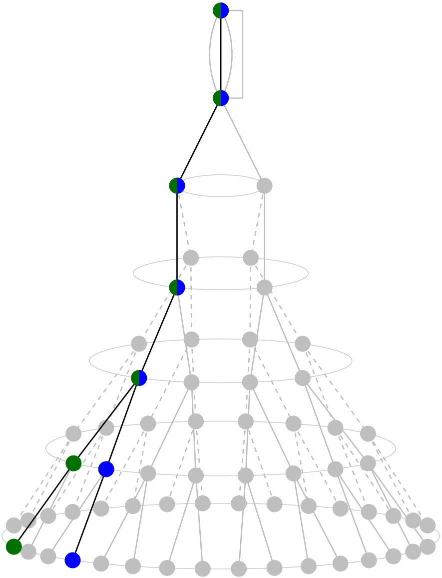

OSIDH PROTOCOL - AN EXAMPLE

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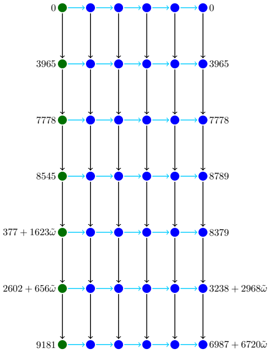

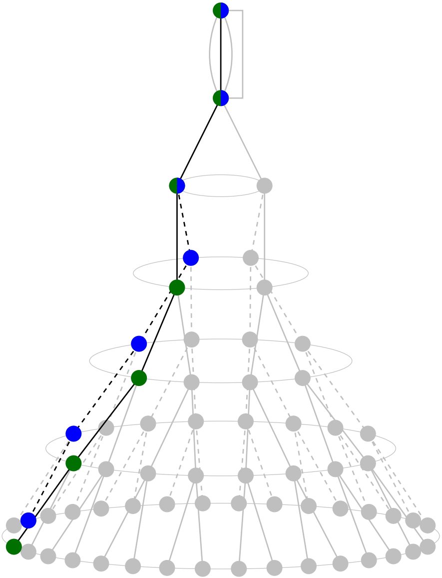

OSIDH PROTOCOL - AN EXAMPLE

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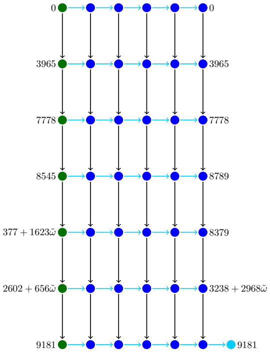

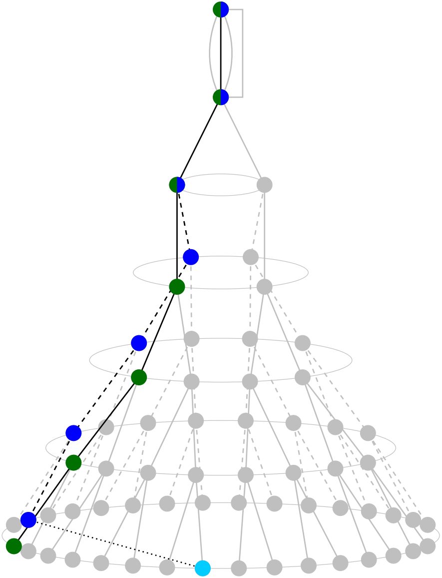

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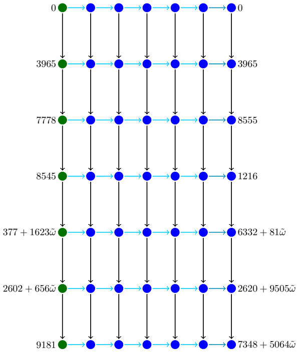
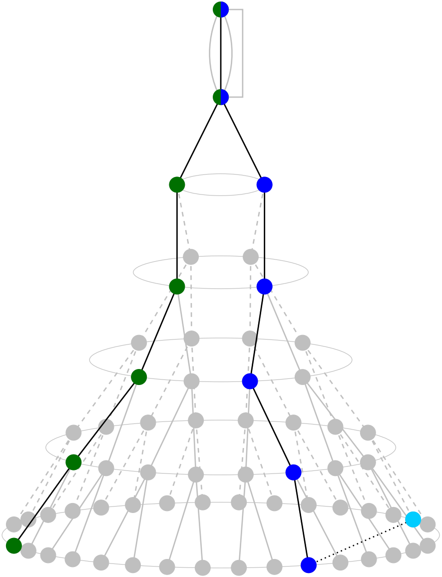
Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$

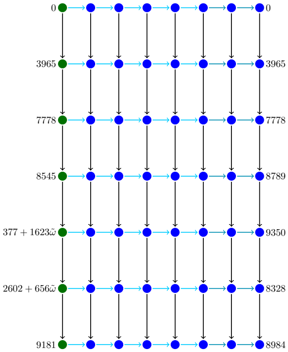


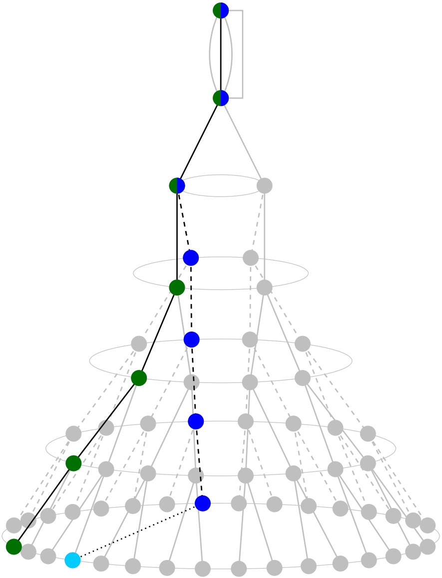


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$





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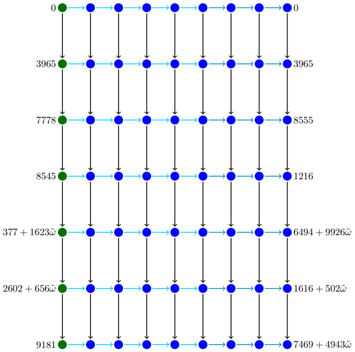

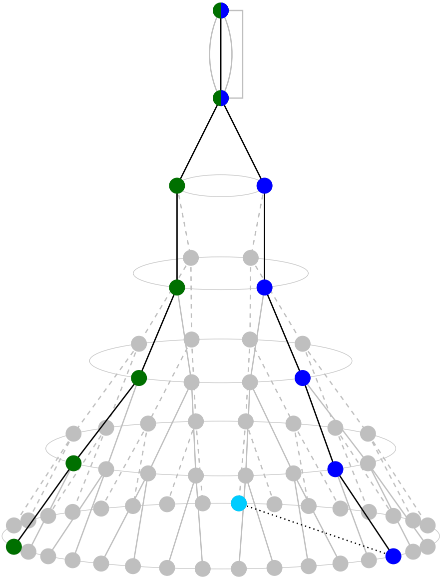

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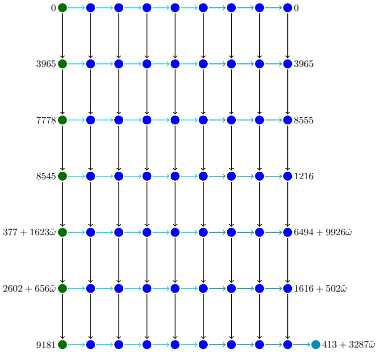

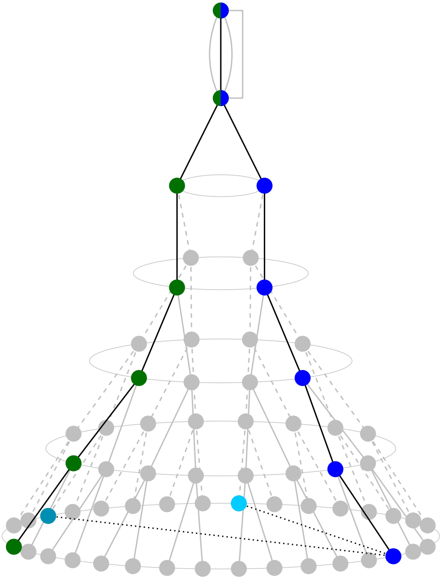

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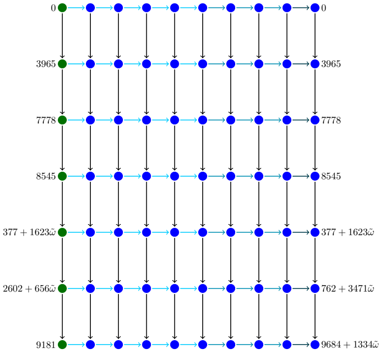

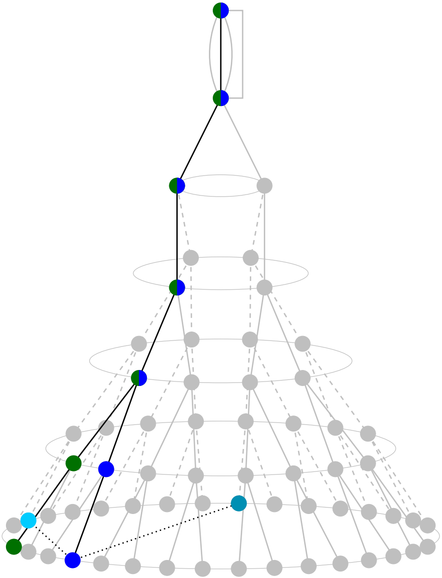

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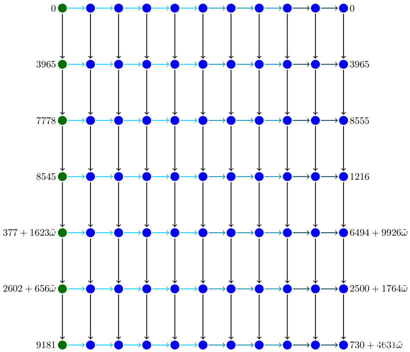
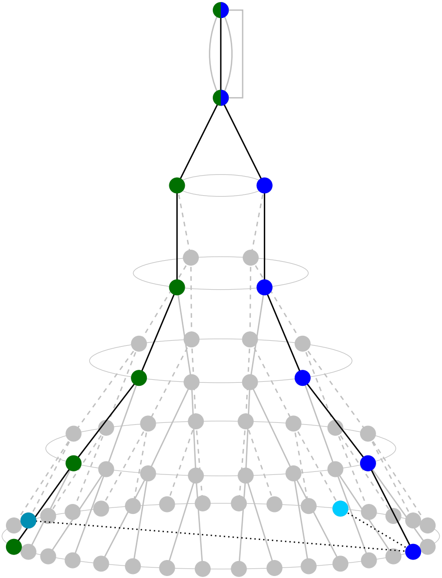
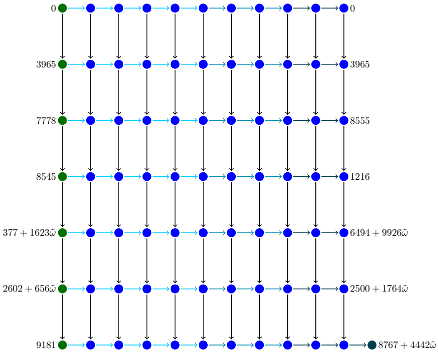
Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$

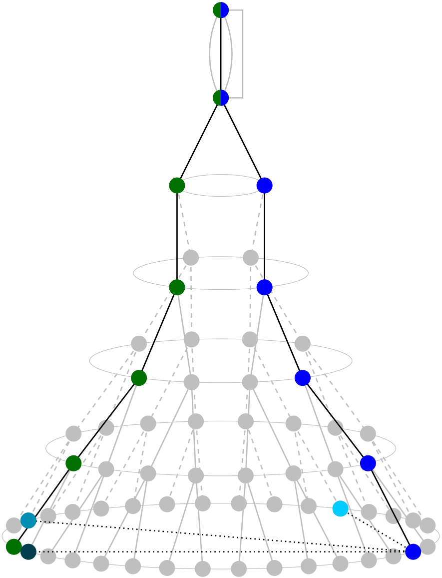




OSIDH PROTOCOL - AN EXAMPLE

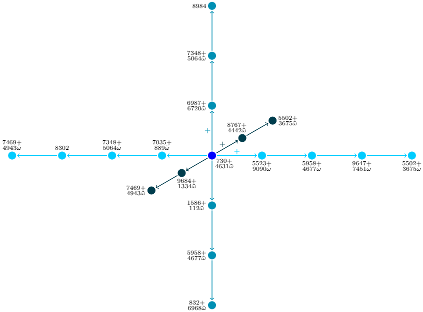
Alice secret key: $\mathfrak{l}_1^5\,\mathfrak{l}_2^3\,\mathfrak{l}_3^2$

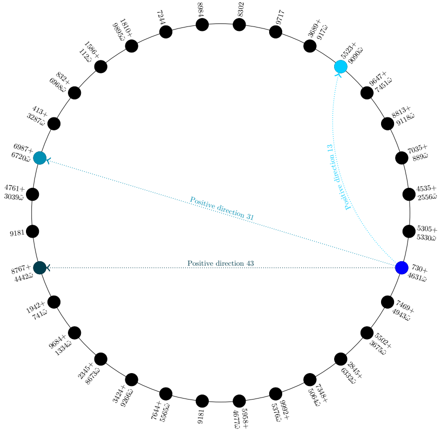




OSIDH PROTOCOL - AN EXAMPLE

Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

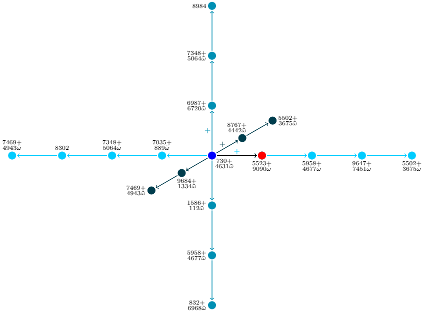


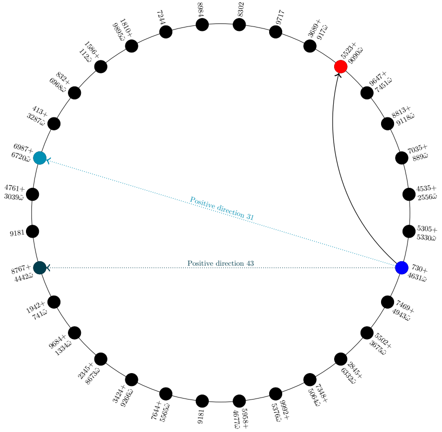


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

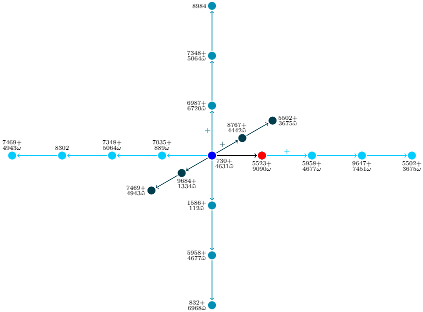


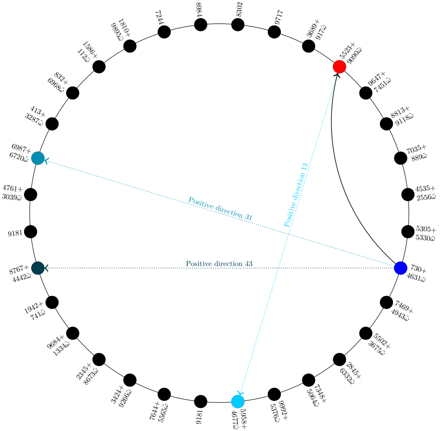


OSIDH PROTOCOL – AN EXAMPLE

L.COLÒ UWAT

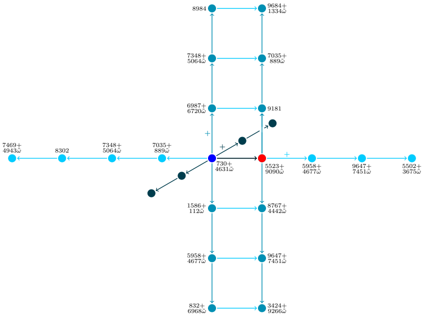
Bob secret key: $\mathfrak{l}_1^3\mathfrak{l}_2\mathfrak{l}_3^2$

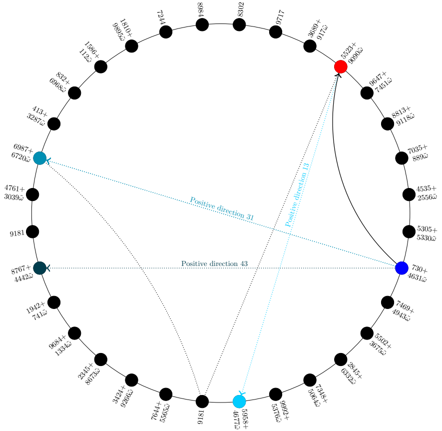




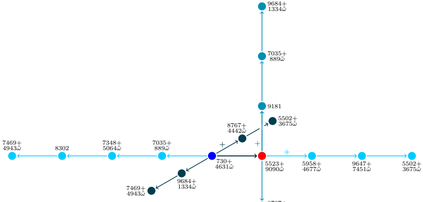
OSIDH PROTOCOL - AN EXAMPLE

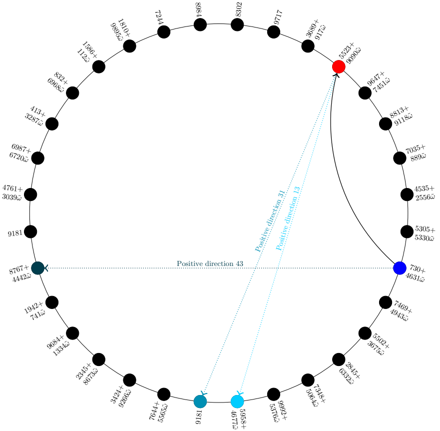
Bob secret key: $\mathfrak{l}_1^3\,\mathfrak{l}_2\,\mathfrak{l}_3^2$





Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

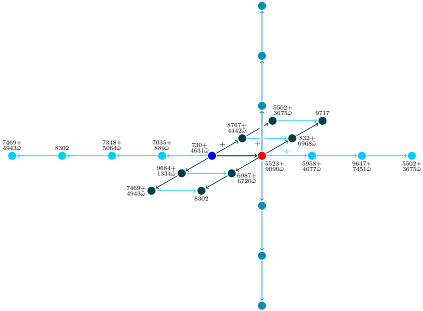


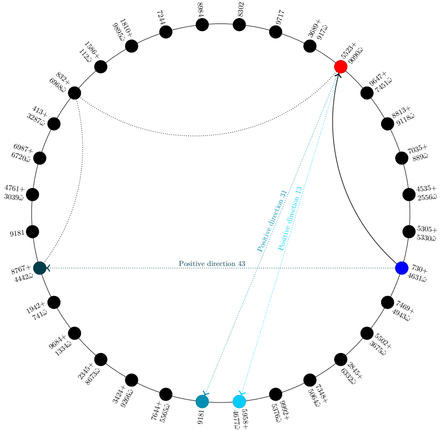


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

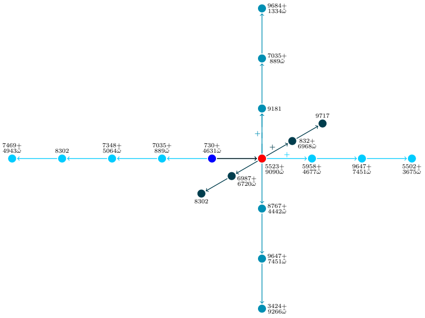
Bob secret key: $\mathfrak{l}_1^3\mathfrak{l}_2\mathfrak{l}_3^2$

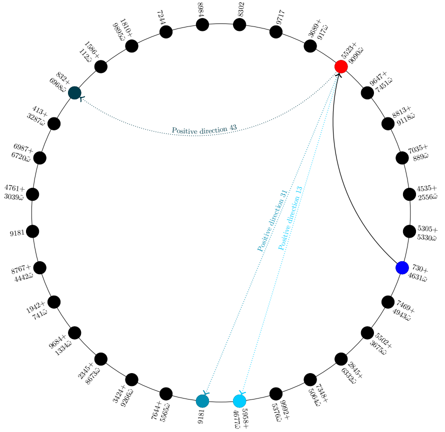




L.COLÒ UWAT

Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

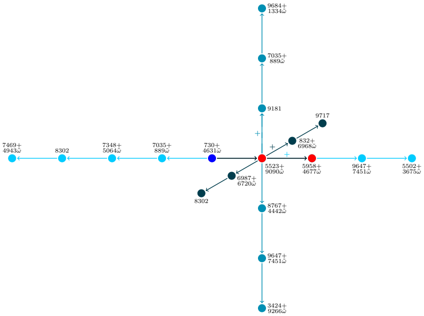


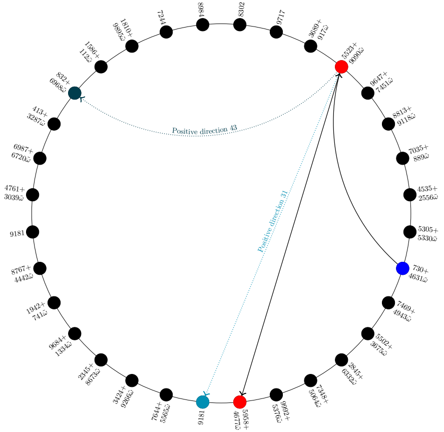


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

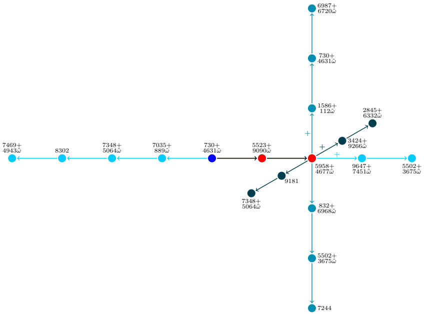
Bob secret key: $\mathfrak{l}_1^3\,\mathfrak{l}_2\,\mathfrak{l}_3^2$

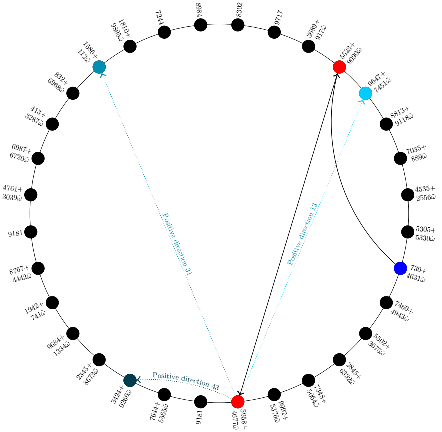




L.COLÒ UWAT

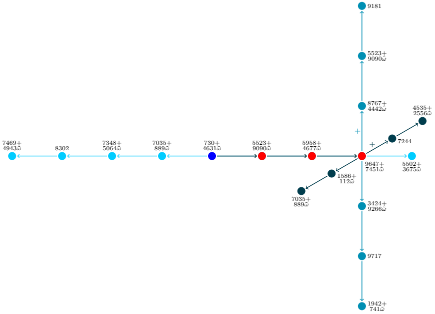
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

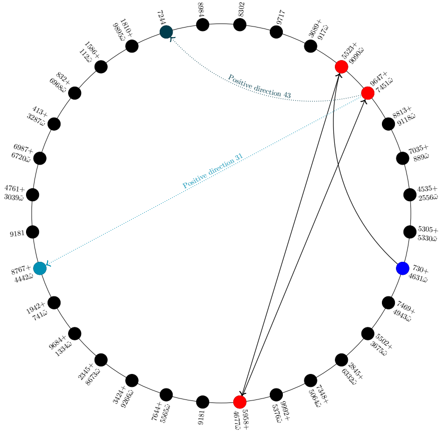




OSIDH PROTOCOL - AN EXAMPLE

Bob secret key: $\mathfrak{l}_1^3\mathfrak{l}_2\mathfrak{l}_3^2$





OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

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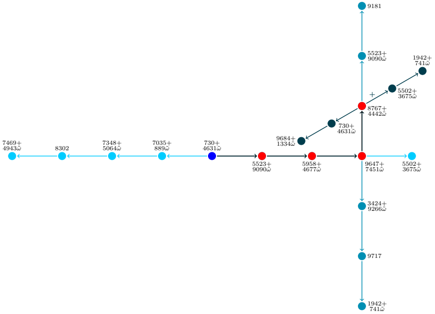
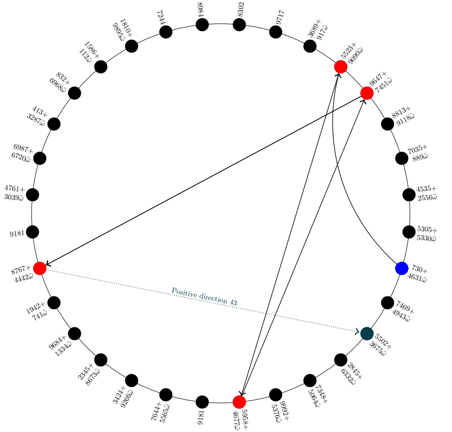
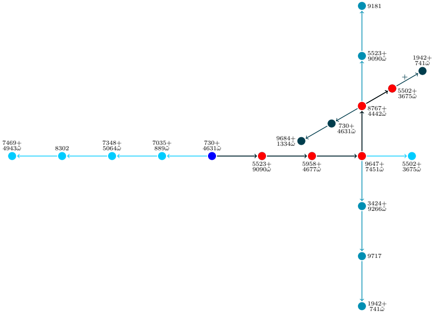
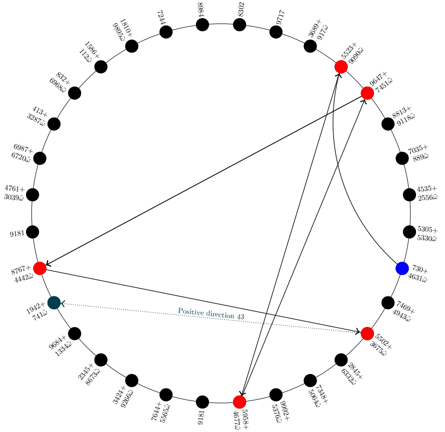




OSIDH PROTOCOL - AN EXAMPLE

Bob secret key: $\mathfrak{l}_1^3\mathfrak{l}_2\mathfrak{l}_3^2$





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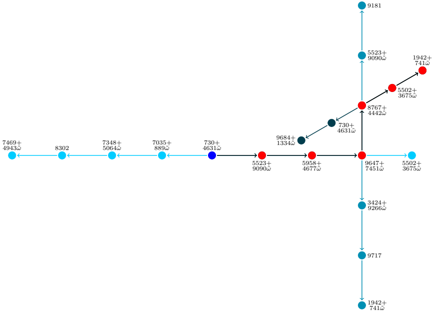
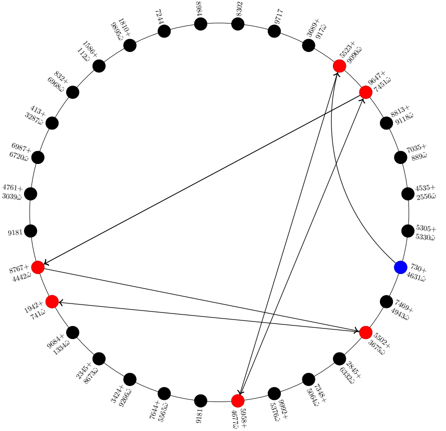
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$





OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

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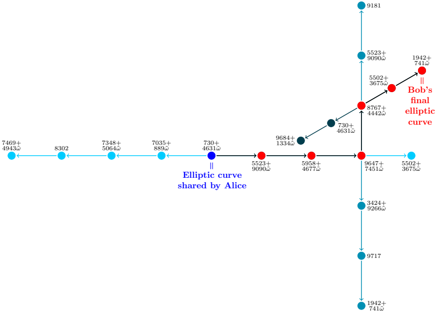

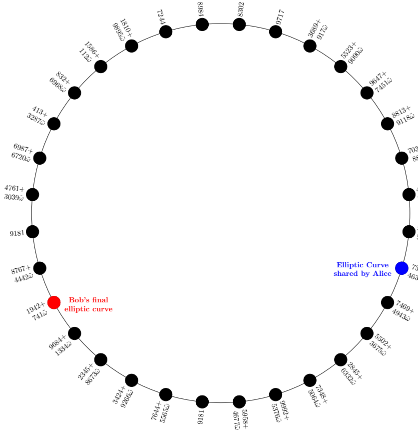

OSIDH PROTOCOL - AN EXAMPLE

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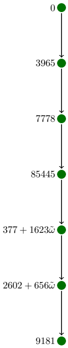
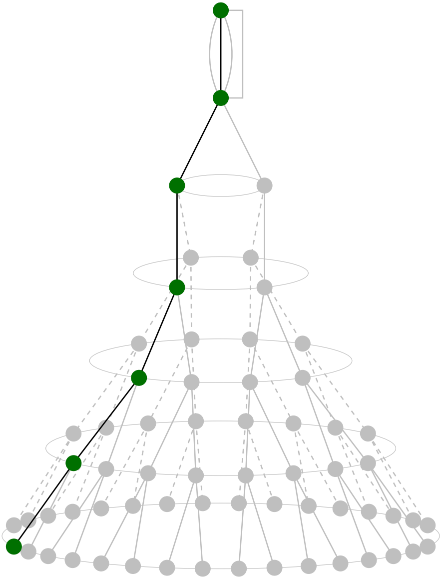
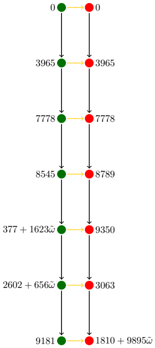
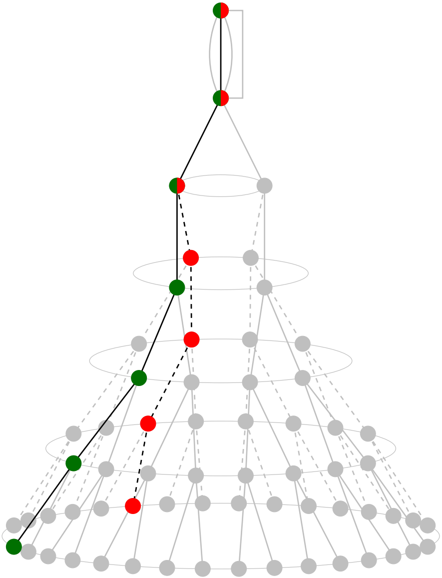




OSIDH PROTOCOL - AN EXAMPLE

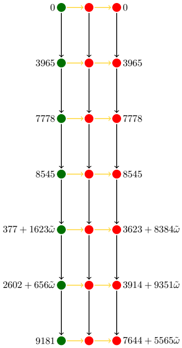
Bob secret key: $\mathfrak{l}_1^3\mathfrak{l}_2\mathfrak{l}_3^2$

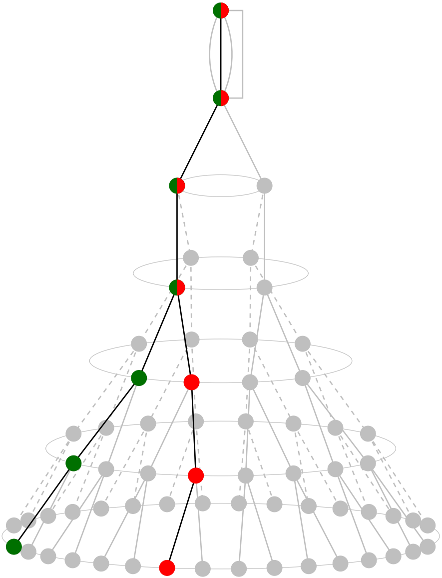




OSIDH PROTOCOL - AN EXAMPLE

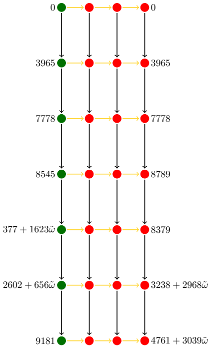
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

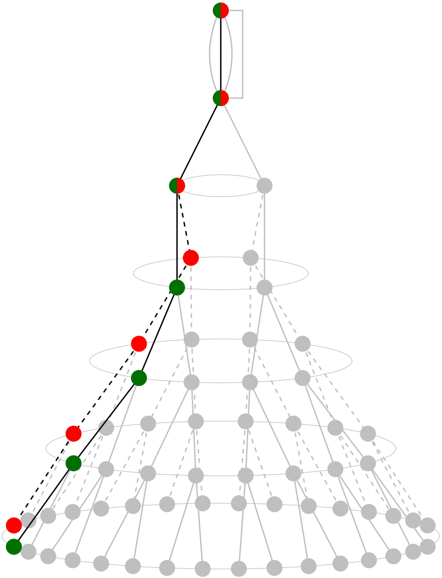




OSIDH PROTOCOL - AN EXAMPLE

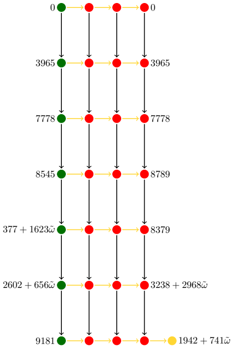
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

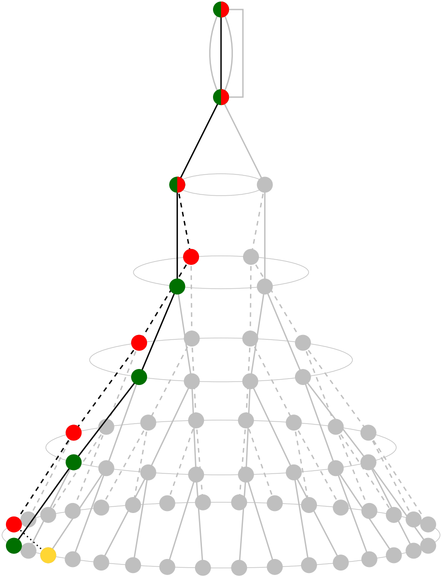




OSIDH PROTOCOL - AN EXAMPLE

Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$





OSIDH PROTOCOL - AN EXAMPLE

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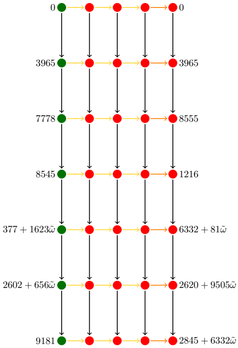
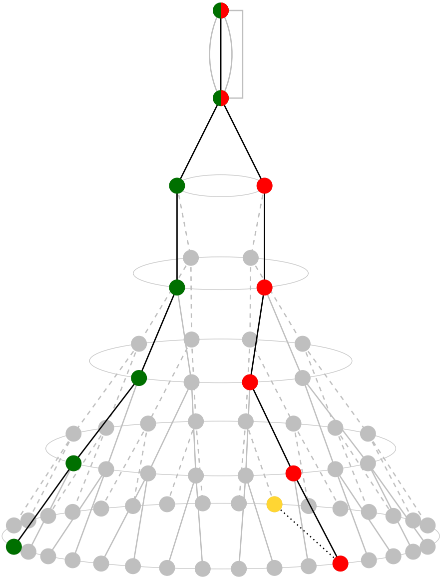
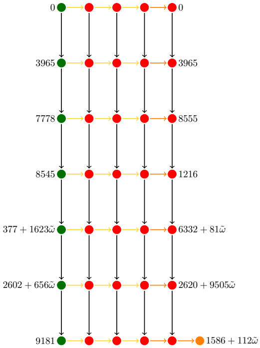


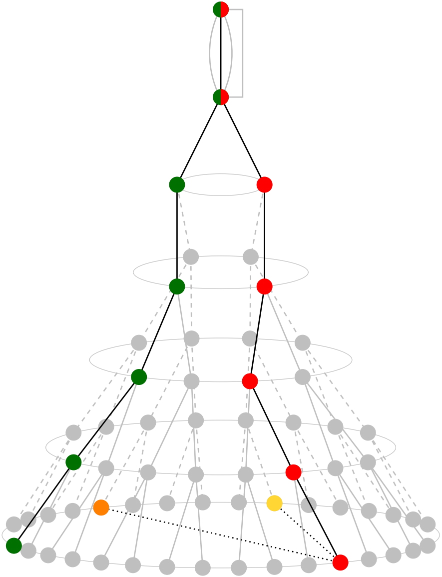


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

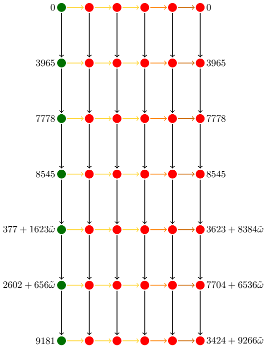
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

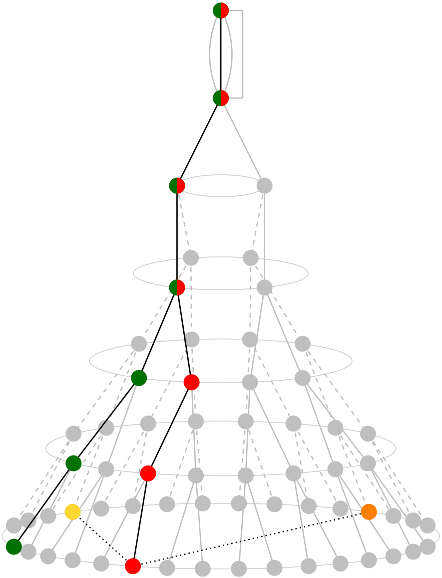




OSIDH PROTOCOL - AN EXAMPLE

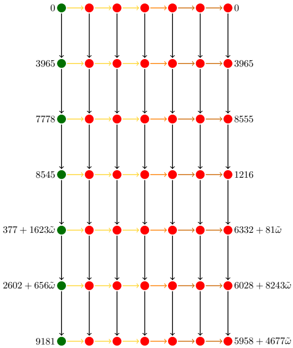
Bob secret key: $\mathfrak{l}_1^3\mathfrak{l}_2\mathfrak{l}_3^2$

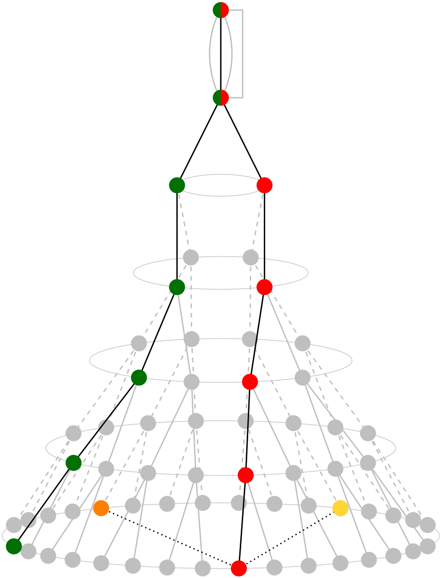




OSIDH PROTOCOL - AN EXAMPLE

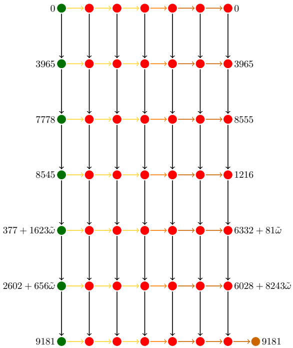
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

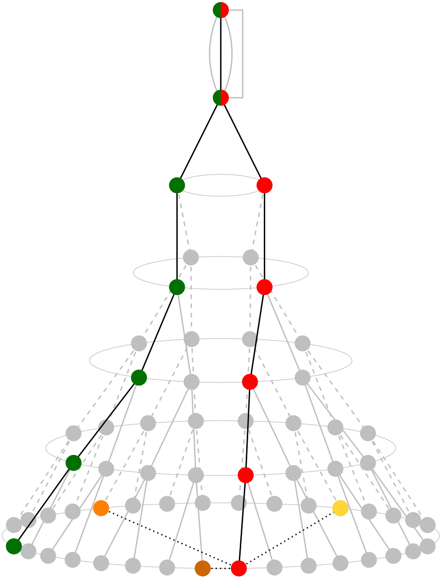




OSIDH PROTOCOL - AN EXAMPLE

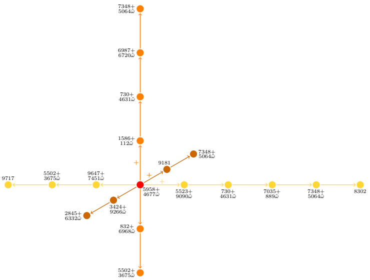
Bob secret key: $\mathfrak{l}_1^3 \mathfrak{l}_2 \mathfrak{l}_3^2$

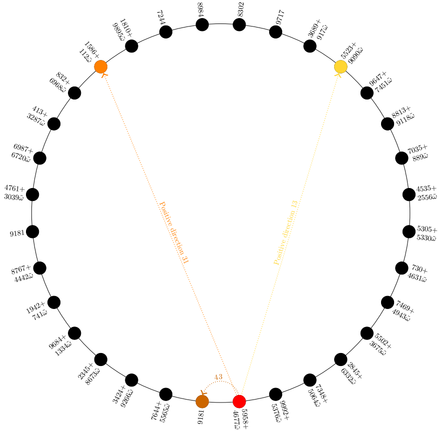




OSIDH PROTOCOL - AN EXAMPLE

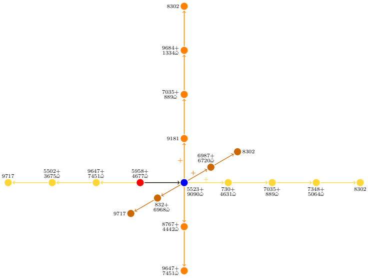
Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$

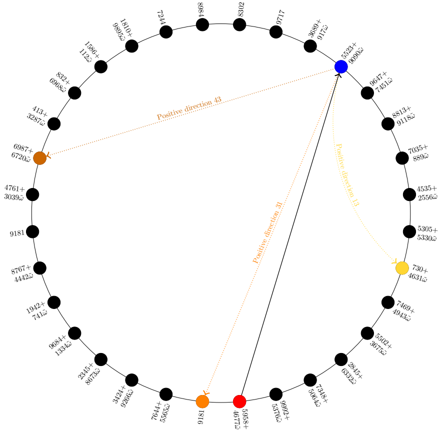




OSIDH PROTOCOL - AN EXAMPLE

Alice secret key: $\mathfrak{l}_1^5\mathfrak{l}_2^3\mathfrak{l}_3^2$

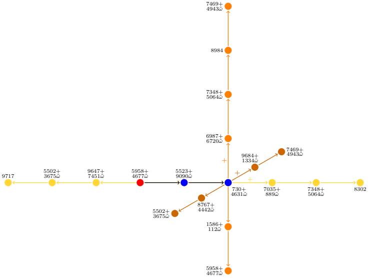


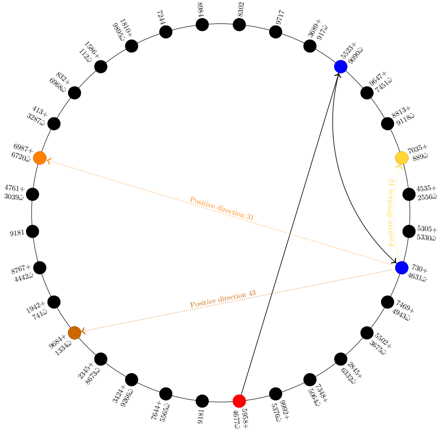


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$





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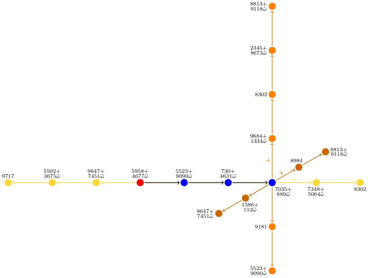
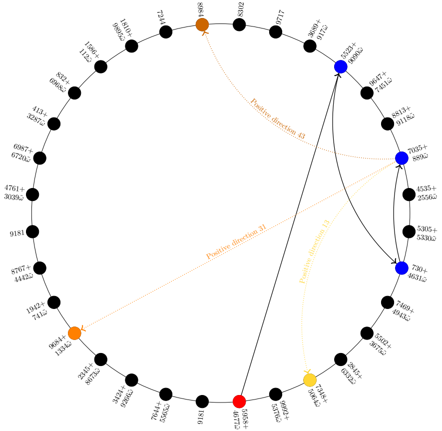
Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$





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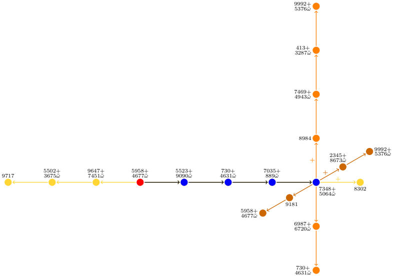

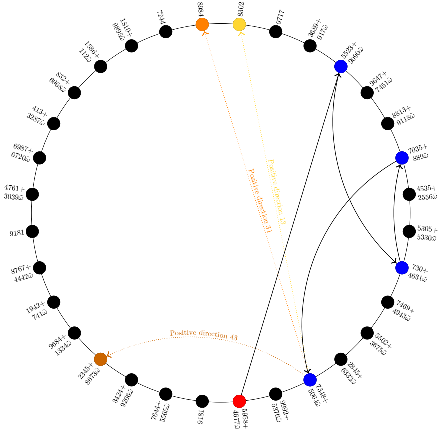

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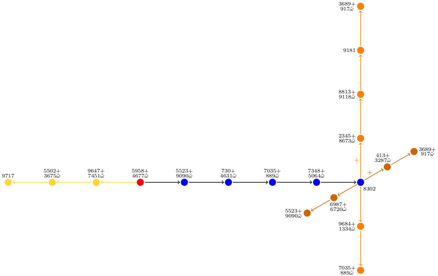
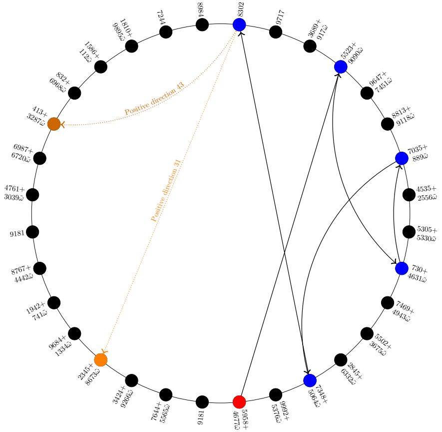
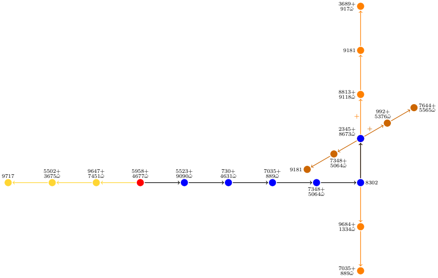
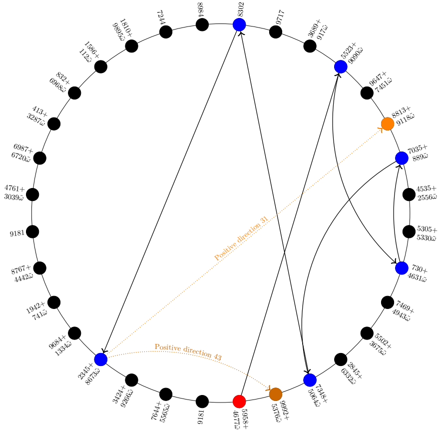




OSIDH PROTOCOL - AN EXAMPLE

Alice secret key: $\mathfrak{l}_1^5 \mathfrak{l}_2^3 \mathfrak{l}_3^2$





OSIDH PROTOCOL - AN EXAMPLE

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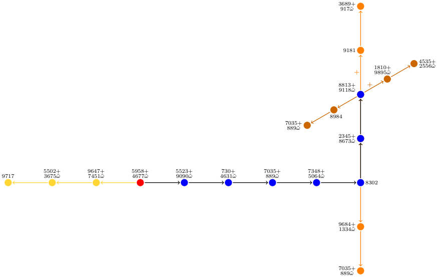

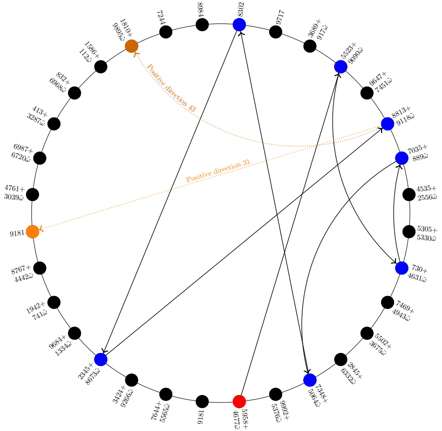

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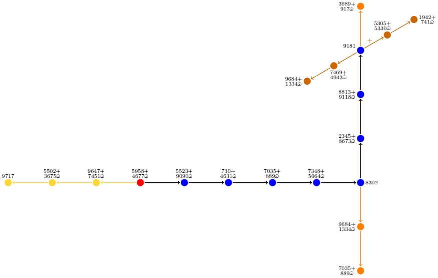
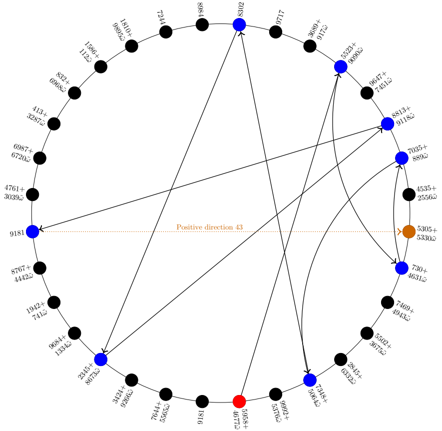
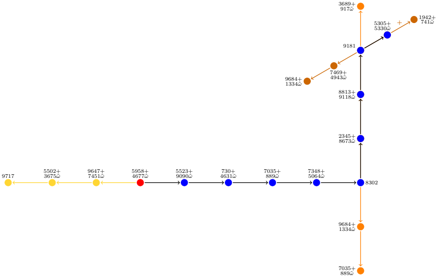


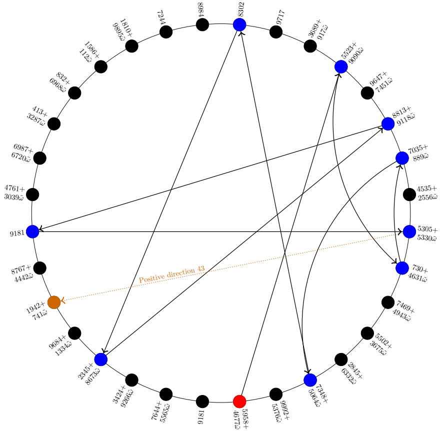


OSIDH PROTOCOL - AN EXAMPLE

L.COLÒ UWAT

Alice secret key: $\mathfrak{l}_1^5\,\mathfrak{l}_2^3\,\mathfrak{l}_3^2$





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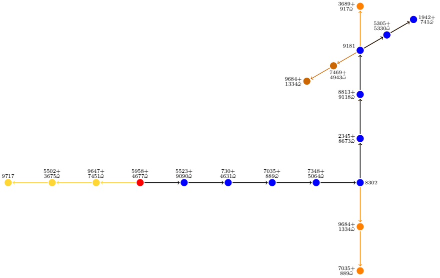

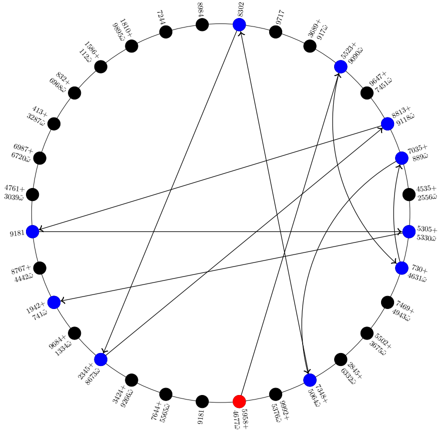

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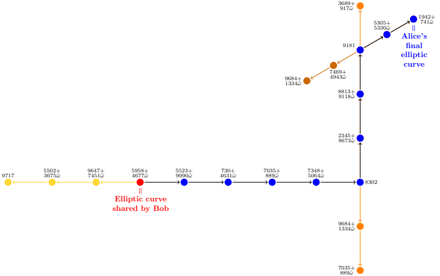
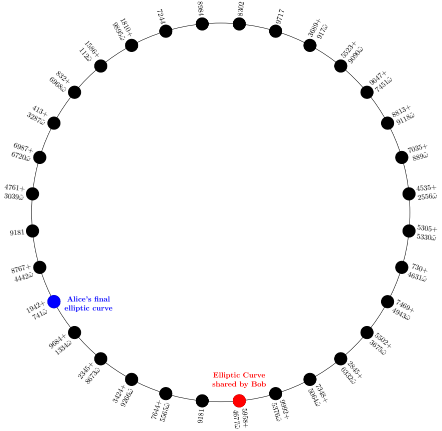
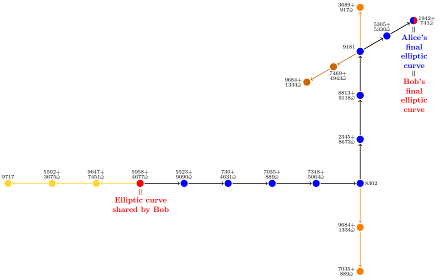
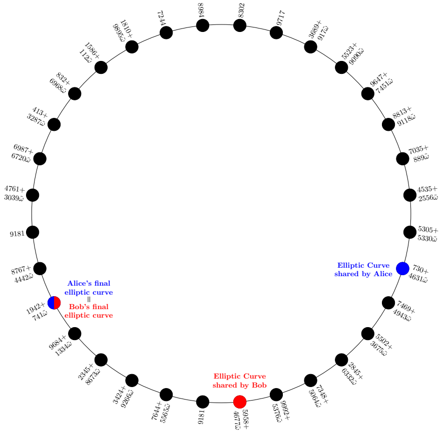




OSIDH PROTOCOL - AN EXAMPLE





SECURITY CONSIDERATIONS

OSIDH PROTOCOL - SECURITY CONSIDERATIONS

For an order $\mathcal{O}$ of conductor $\ell^n M$, we note that $\mathcal{C}\ell(\mathcal{O}) \simeq \mathrm{SS}_{\mathcal{O}}^{pr}(p)$ and define

$$I = I_1 \times \ldots \times I_t \subseteq \mathbb{Z}^t \quad \text{where } I_j = [-r_j, r_j].$$

The security of OSIDH depends on the following maps

$$I = \prod_{i=1}^{t} [-r_i, r_i] \longrightarrow \mathrm{SS}_{\mathcal{O}}^{pr}(p) \longrightarrow \mathrm{SS}(p)$$

We deal with the problem of covering a reasonable number of curves in (p).

Supersingular covering bound

We say that the map $\mathcal{C}\ell(\mathcal{O}) \simeq \mathrm{SS}_{\mathcal{O}}^{pr}(p) \longrightarrow \mathrm{SS}(p)$ is λ-*surjective* if

$$p^{\lambda} \leq \#\mathcal{C}\ell(\mathcal{O})$$

where λ is the *logarithmic covering radius*. We get

$$\lambda \log_{\ell}(p) \leq n + \log_{\ell}(M) + \log_{\ell}(h(\mathcal{O}_K))$$

OSIDH PROTOCOL - SECURITY CONSIDERATIONS

For an order $\mathcal{O}$ of conductor $\ell^n M$, we note that $\mathcal{C}\ell(\mathcal{O}) \simeq \mathrm{SS}_{\mathcal{O}}^{pr}(\rho)$ and define

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The security of OSIDH depends on the following maps

$$I = \prod_{i=1}^{t} [-r_i, r_i] \longrightarrow \mathrm{SS}_{\mathcal{O}}^{pr}(\rho) \longrightarrow \mathrm{SS}(p)$$

Supersingular injectivity bound

How can one insure the injectivity of the map $\mathrm{SS}_{\mathcal{O}}^{pr}(\rho) \to \mathrm{SS}(p)$? We set

$$n + \log_\ell(M) + \frac{1}{2}\log_\ell(|\Delta_K|) \leq \frac{1}{2}\log_\ell(p)$$

If (SIB) holds, then the map $\mathrm{SS}_{\mathcal{O}}^{pr}(\rho) \to (p)$ is injective.

OSIDH PROTOCOL - SECURITY CONSIDERATIONS

L.COLÒ UWAT

For an order $\mathcal{O}$ of conductor $\ell^n M$, we note that $\mathcal{C}\ell(\mathcal{O}) \simeq \mathrm{SS}^{pr}_{\mathcal{O}}(\rho)$ and define

$$I = I_1 \times \ldots \times I_t \subseteq \mathbb{Z}^t \quad \text{where } I_j = [-r_j, r_j].$$

The security of OSIDH depends on the following maps

$$I = \prod_{i=1}^{t} [-r_i, r_i] \longrightarrow \mathrm{SS}^{pr}_{\mathcal{O}}(\rho) \longrightarrow \mathrm{SS}(p)$$

Class group covering bound

In order to have a uniform element of $\mathcal{C}\ell(\mathcal{O})$ it is desirable to be able to reach all elements of $\mathcal{C}\ell(\mathcal{O})$.

$$\sum_{i=1}^{t} \log_\ell(2r_i + 1) \geq \lambda \left(n + \log_\ell(M) + \log_\ell(h(\mathcal{O}_K))\right)$$

For an order $\mathcal{O}$ of conductor $\ell^n M$, we note that $\mathcal{C}\ell(\mathcal{O}) \simeq \mathrm{SS}^{pr}_{\mathcal{O}}(p)$ and define

$$I = I_1 \times \ldots \times I_t \subseteq \mathbb{Z}^t \quad \text{where } I_j = [-r_j, r_j].$$

The security of OSIDH depends on the following maps

$$I = \prod_{i=1}^{t} [-r_i, r_i] \longrightarrow \mathrm{SS}^{pr}_{\mathcal{O}}(p) \longrightarrow \mathrm{SS}(p)$$

Minkowski norm bound

The set of elements obtained by random walks should contain no cycle; thus,

$$\sum_{i=1}^{t} r_i \log_\ell(q_i) \leq n + \log_\ell(M) + \frac{1}{2}\log_\ell(|\Delta_K|/4)$$

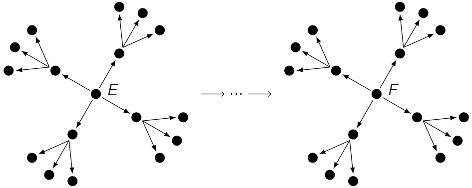
The attack of Dartois and De Feo exploits the non-injectivity of the map $I \to \mathrm{SS}^{pr}_{\mathcal{O}}(p)$ to recover an endomorphism of E.

Key generation

On one side, A begins with $F = E$.

- Split primes: for each prime q_i in $\mathcal{P}_S$, choose a random $s_i \in I_i$, constructs the q_i-isogeny walk of length s_i while pushing forward the other direction as well as the q-clouds at each prime q in $\mathcal{P}_A$ and $\mathcal{P}_B$.
- Non-split primes: for each prime choose a random walk in the cloud to a new curve F and push forward the remaining unused q-clouds.

The data F and q-isogeny chains at primes q in $\mathcal{P}_s$ and q-clouds at primes q in $\mathcal{P}_B$ constitute A's public key.



PARAMETER SELECTION - AN EXAMPLE

L.COLÒ UWAT

We set $\Delta_K = -3$ and $\ell = 2$.

We begin with $t = 10$ and a bit Bound $B_s = 32$.

Split Primes

	q :	7	13	19	31	37	43	61	67	73	79
$\mathcal{P}_s$:	r :	11	8	7	6	6	6	5	5	5	5
	$\#$:	23	17	15	13	13	13	11	11	11	11

This gives a logarithmic contribution of

$$\sum_{j=1}^{10} \log_2(2r_j + 1) = 37.4569...$$

to the entropy of the random walk.

The logarithmic norm, which we must bound is:

$$\sum_{j=1}^{10} r_j \log_2(q_j) = 306.2115...(< 320 = 32 \cdot 10).$$

PARAMETER SELECTION - AN EXAMPLE

We set $\Delta_K = -3$ and $\ell = 2$.
We begin with $t = 10$ and a bit Bound $B_s = 32$.

Non-Split Primes

We partition the remaining primes up to 163 into sets $\mathcal{P}_A$ and $\mathcal{P}_B$, with a radius for the cloud (or eddy), as follows:

$\mathcal{P}_A$:	q :	2	11	17	41	47	59	83	101	103	109	131	149	151	157
	r :	7	2	1	1	1	1	1	1	1	1	1	1	1	1
	# :	128	132	18	42	48	60	84	102	102	108	132	150	150	156

$\mathcal{P}_B$:	q :	3	5	23	29	53	71	89	97	107	113	127	137	139	163
	r :	4	3	1	1	1	1	1	1	1	1	1	1	1	1
	# :	81	150	24	30	54	72	90	96	108	114	126	138	138	162

Both sets leak the horizontal directions for these primes, giving an additional contribution of ≈ 28 bits to the logarithmic norm.
These prime sets each contribute a $\log_2(M)$ of 90 bits, such that n must be at least 244 to defeat the lattice-based class group attack.

PARAMETER SELECTION - CONCLUSION

The norm bound suggests using a uniform bound B_s on $r_j \log_\ell(q_j)$ rather than the exponents r_j. This gives

$$\lambda \log_\ell(p) \leq \sum_{i=1}^{t} \log_\ell(2r_j + 1) \leq \sum_{j=1}^{t} r_j \log_\ell(q_j) \leq tB_s < n + \log_\ell(M)$$

for which ($t = 64$, $B_s = 16$, $n = 1024$) represent a choice of parameters ensuring injectivity of $I \to \mathcal{C}\ell(\mathcal{O})$.

THANK YOU FOR YOUR ATTENTION