COMPLETIONS AND EXTENSIONS

A Computation

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Contents

1 Topology

1.1 Product Topology

The Cartesian product space

$$X_p = \prod_{n\geq 0} \{0, 1, \ldots, p-1\} = \{0, 1, \ldots, p-1\}^{\mathbb{N}}$$

can be considered as a topological space, with respect to the product topology of the finite discrete sets $\{0, 1, \ldots, p-1\}$.

By the Tychonof theorem, X_p is compact. It is also totally disconnected, i.e., the connected components are points.

Let us recall that the discrete topology can be defined by a metric

$$\delta(a, b) = 1 - \delta_{a,b} = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

Several metrics compatible with the product topology on X_p can be deduced form these discrete ones. For instance, for $a = (a_0, a_1, a_2, a_3, \ldots)$ and $a = (b_0, b_1, b_2, b_3, \ldots)$ in X_p we can define

$$d(x, y) = \sup_{n\geq 0} \frac{\delta(a_n, b_n)}{p^n}$$

$$d'(x, y) = \sum_{n\geq 0} \frac{\delta(a_n, b_n)}{p^{n+1}}$$

and so on.

Although all metrics on a compact metrizable space are uniformly equivalent, they are not equally interesting. We will chose the first one defined above since it will give a more faithful image of the coset structure of $\mathbb{Z}_p$.

For each integer $n \in \mathbb{N}$, all cosets of $p^n\mathbb{Z}_p$ in $\mathbb{Z}_p$ should be isometric (and, in particular, have the same diameter).

1.2 Topological Groups

Definition. A topological group is a group G equipped with a topology such that the map $G \times G \longrightarrow G$ defined as $(x, y) \to xy^{-1}$ is continuous.

Remark. With addition, $\mathbb{Z}_p$ is a topological group. We have indeed

$$a' \in a + p^n\mathbb{Z}_p \ , \quad b' \in b + p^n\mathbb{Z}_p \implies a' - b' \in a - b + p^n\mathbb{Z}_p$$

for all $n > 0$. In other words, using the p-adic metric defined above we have

$$|x - a|_p \leq |p^n|_p = p^{-n} \ , \quad |y - b|_p \leq |p^n|_p = p^{-n} \implies |(x - y) - (a - b)|_p \leq p^{-n}$$

proving the continuity of the map $(x, y) \to x - y$ at any point (a, b).

Remark. With respect to multiplication, $\mathbb{Z}_p^\times$ is a topological group. There is a fundamental system of neighborhoods of its neutral element 1 consisting of subgroups:

$$1 + p\mathbb{Z}_p \supseteq 1 + p^2\mathbb{Z}_p \supseteq \ldots \supseteq 1 + p^n\mathbb{Z}_p \supseteq \ldots$$

consists of subgroups: if $\alpha, \beta \in \mathbb{Z}_p$we see that $(1 + p^n\beta)^{-1} = 1 + p^n\beta'$ for some $\beta' \in \mathbb{Z}_p$ and hence

$$a = 1 + p^n\alpha \ , \quad b = 1 + p^n\beta \implies ab^{-1} = (1 + p^n\alpha)(1 + p^n\beta') = 1 + p^n\gamma$$

for some $\gamma \in \mathbb{Z}_p$. Consequently,

$$a' \in a(1 + p^n\mathbb{Z}_p) \ , \quad b' \in b(1 + p^n\mathbb{Z}_p) \implies a'b'^{-1} \in ab^{-1}(1 + p^n\mathbb{Z}_p) \quad (n \geq 1)$$

and $(x, y) \to xy^{-1}$ is continuous. It can be shown that $1 + p\mathbb{Z}_p$ is a subgroup of index $p - 1$ in $\mathbb{Z}_p^\times$. It is also open by definition. With respect to multiplication,all subgroups $1 + p^n\mathbb{Z}_p$ $(n > 1)$ are topological groups.

1.3 Topological Rings

Definition. A topological ring A is a ring equipped with a topology such that the maps

$$(x, y) \to x + y : A \times A \to A$$

$$(x, y) \to x \cdot y : A \times A \to A$$

are continuous.

Remark. If A is a topological ring, the subgroup $A^\times$ of units is not, in general, a topological group, since $x \to x^{-1}$ is not necessarily continuous for the induced topology. However, we can consider the embedding

$$x \to (x, x^{-1}) : A \to A \times A$$

and give $A^\times$ the initial topology: it is finer than the topology induced by A. For this topology, $A^\times$ is a topological group: the continuity of the inverse map, induced by the symmetry $(x, y) \to (y, x)$ of $A \times A$ is now obvious.

Proposition 1.1. *With the p-adic metric the ring $\mathbb{Z}_p$ is a topological ring. It is a compact, complete, metrizable space.*

1.4 Inverse Limits

When a projective system $(E_n, \varphi_n)_{n\geq 0}$ is formed of topological spaces and continuous transition maps, the construction of the projective limit shows immediately that the projective limit (E, ψ_n) is a topological space equipped with continuous maps $\psi_n : E \to E_n$ having the universal property with respect to continuous maps.

We know that $\mathbb{Z}_p$ is an inverse limit

$$\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}$$

Proposition 1.2. *In a projective limit $E = \lim_{\leftarrow} E_n$ of topological spaces, a basis of the topology is given by the sets $\psi_n{-}1(\mathcal{U}_n)$ where $n \geq 0$ and $\mathcal{U}_n$ is an arbitrary open set in E_n.*

1.5 Metric Spaces

Both $\mathbb{R}$ and $\mathbb{Q}_p$ are normed fields and complete metric spaces, both are completions of $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in both of them, they are separable.

Since we have an absolute value on $\mathbb{Q}_p$, we can define a metric from it by $d_p(x, y) = |x - y|_p$. As usual we define an open ball in $\mathbb{Q}_p$ with center a and radius r to be

$$B(a, r) = \{x \in \mathbb{Q}_p \mid d_p(a, x) < r\} = \left\{x \in \mathbb{Q}_p \mid \ |a - x|_p < r\right\}$$

The close ball in $\mathbb{Q}_p$ with center a and radius r is denoted by

$$\overline{B}(a, r) = \left\{x \in \mathbb{Q}_p \mid \ |a - x|_p \leq r\right\}$$

Finally, the sphere with center a and radius r is denoted by

$$S(a, r) = \left\{x \in \mathbb{Q}_p \mid \ |a - x|_p = r\right\}$$

Remark. Since

$$\{|x - y| \mid x, y \in \mathbb{Q}_p\} = \{p^n \mid n \in \mathbb{Z}\} \cup \{0\}$$

we only need to consider the balls of radi $r = p^n$, where $n \in \mathbb{Z}$.

Proposition 1.3. *The following properties hold*

- *The open and close balls $B(a, r)$ and $\overline{B}(a, r)$ are both an open and close sets in $\mathbb{Q}_p$.*
- *The sphere $S(a, r)$ is both an open and close set in $\mathbb{Q}_p$.*
- *Any point of a ball is its center, i.e., for every $b \in B(a, r)$, $B(b, r) = B(a, r)$ (the same is true for closed balls).*
- *Any two balls in $\mathbb{Q}_p$ have non empty intersection if and only if one is contained in the other.*

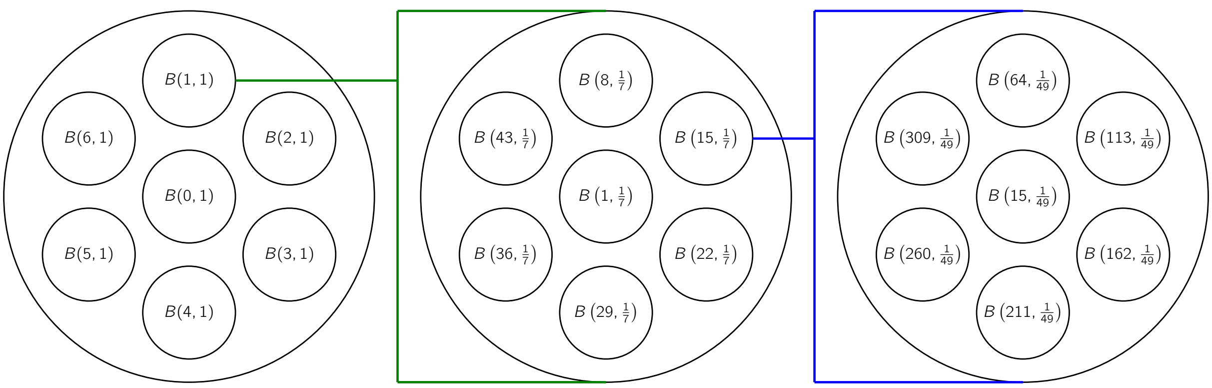

Figure 1: A model for $\mathbb{Z}_7$

Taking $\mathbb{Z}_7$ as example we see that $\mathbb{Z}_p = \bigcup_{x\in\{0,\dots,6\}} B(x, 1)$ in addition it holds that

$$B(x, 7^{-k}) = \bigcup_{j\in\{0,\dots,6\}} B(x + j \cdot 7k + 1, 7^{k+1})$$

Proposition 1.4. *The set of all balls in $\mathbb{Q}_p$ is countable.*

Theorem 1.5. *The set $\mathbb{Z}_p$ is compact and the space $\mathbb{Q}_p$ is locally compact.*

Theorem 1.6. *The space $\mathbb{Q}_p$ is totally disconnected.*

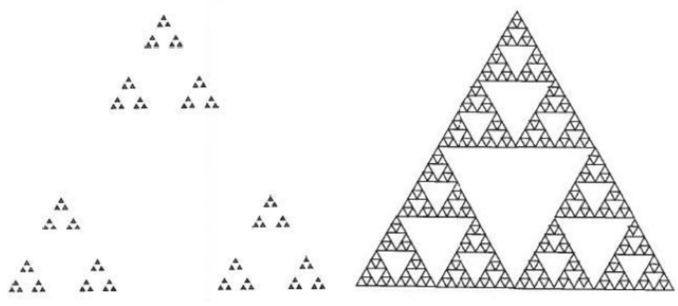

Figure 2: Model of $\mathbb{Z}_3$ and Sierpinski gasket.

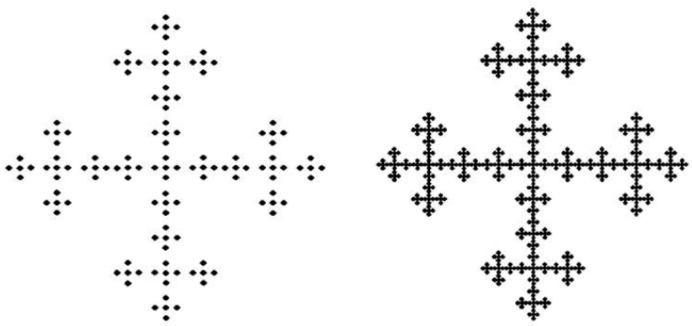

Figure 3: Fractal Model for $\mathbb{Z}_5$.

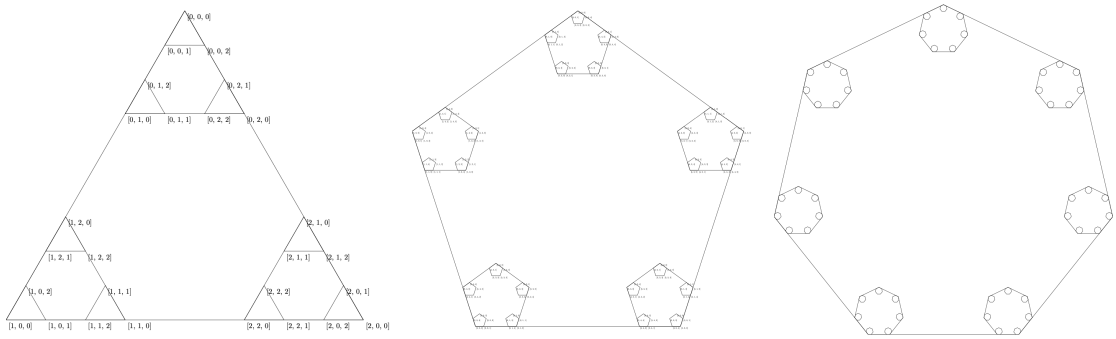

Figure 4: Level-2 models for $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$

2 Completions

2.1 Construction

Definition. A valued field $(K, |\ |)$ is called complete if every Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$ in K converges to an element a of K:

$$\lim_{n\to+\infty} |a_n - a| = 0$$

as usual $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{for every } n, m \geq N$$

From any valued field $(K, |\ |)$ we get a complete value field $(\hat{K}, |\ |)$ by the process of completion. We denote

$$\mathcal{C} = \{\text{set of Cauchy sequences of } (K, |\ |)\}$$

$$\mathfrak{m} = \{\text{set of null-sequences of } (K, |\ |)\} = \{\text{Cauchy sequences of } (K, |\ |) \text{ that tend to } 0\}$$

It can be easily proved that $\mathcal{C}$ is a ring and $\mathfrak{m}$ is a maximal ideal. We can therefore define a field $\hat{K} = \mathcal{C}/\mathfrak{m}$ and we construct an embedding

$$\begin{aligned} K &\hookrightarrow \hat{K} \\ a &\rightarrow [\{a\}_{n\in\mathbb{N}}] \end{aligned}$$

The valuation $|\ |$ is extended from K to $\hat{K}$ in the following way: if $a \in K$ is represented by the sequence $\{a_n\}_{n\in\mathbb{N}}$, then $|a| = \lim\limits_{n\to+\infty} |a_n|$.

Lemma 2.1. *The limit exists.*

Proof. If $a = 0$, then $\{a_n\}_{n\in\mathbb{N}} \in \mathfrak{m}$, so that $a_n \to 0$, so that $|a_n| \to 0$ and so $|a| = 0$ which is only reasonable. On the other hand, if $a \neq 0$ then Lemma 2.2 says that the sequence $|a_n|$ is constant for sufficiently large n, which means it certainly has a limit. □

Lemma 2.2. *Let $\{a_n\}_{n\in\mathbb{N}} \in \mathcal{C}$. The sequence of real numbers $\{|a_n|\}_{n\in\mathbb{N}}$ is eventually stationary, that is, there exits an integer N such that $|a_n| = |a_{n+1}|$ whenever $n \geq N$.*

Proof.

- We know that $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy and it does not tend to 0, then there exist $c > 0$ and $N_1 \in \mathbb{N}$ such that $|a_n| \geq c > 0$ for every $n \geq N_1$.
- By definition there exists N_2 such that $|x_n - x_m| < c$.
- Now $N = \max N_1, N_2$ and

  $$|x_n - x_m| < |x_n|, |x_m| \leq \max\{|x_n|, |x_m|\} \quad \forall n, m \geq N$$

  thus, $|x_n| = |x_m|$ by Lemma 2.3.

□

Lemma 2.3. *If $|x| \neq |y|$, then $|x + y| = \max\{|x|, |y|\}$.*

Proof. Suppose $|x| > |y|$, then $|x + y| \leq |x|$. On the other hand $x = (x + y) - y$ and so $|x| \leq \max\{|x + y|, |y|\}$ but $|x| > |y| \Rightarrow |x| \leq |x + y|$. □

Proposition 2.4. *The following hold*

- *for $a \in \hat{K}$, $|a|$ does not depend on the choice of the sequence $\{a_n\}_{n\in\mathbb{N}}$ defining a.*
- *Let $a \in \hat{K}$, $|x| = 0$ if and only if $x = 0$.*
- *$(\hat{K}, |\ |)$ is non-archimedean.*
- *For $a \in K$, $|a|_K = |a|_{\hat{K}}$.*

2.2 Properties

To prove that we have indeed obtained a completion we have to show that K is dense in $\hat{K}$ and that $\hat{K}$ is complete.

Proposition 2.5. *K is dense in $\hat{K}$.*

Proof. We need to show that any open ball around an element $a \in \hat{K}$ contains an element of K (a constant sequence).

Fix $\epsilon > 0$ and a representation $\{a_n\}_{n\in\mathbb{N}}$ of a. Take $\epsilon' < \epsilon$; there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon'$ for every $n, m \geq N$. Now we show that $\{a_N\}_{n\in\mathbb{N}} \in B(a, \epsilon)$. We recall that $a - \{a_N\}$ is represented by $\{a_n - a_N\}_{n\in\mathbb{N}}$

$$|a - a_N| = \lim_{n\to+\infty} |a_n - a_N|$$

and $|a_n - a_N| < \epsilon'$ for any $n \geq N$. Thus,

$$|a - a_N| = \lim_{n\to+\infty} |a_n - a_N| < \epsilon' < \epsilon$$

□

Proposition 2.6. *$\hat{K}$ is complete with respect to $|\ |$.*

Sketch of the Proof. **(i)** $a_1, a_2, \ldots$ Cauchy sequence of elements of $\hat{K}$. Since $K \hookrightarrow \hat{K}$ is dense we know that there exist $a^{(1)}, a^{(2)}, \ldots \in K$ such that

$$\lim_{n\to+\infty} \left| a_n - a^{(n)} \right| = 0$$

(ii) $\{a^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence of elements in K and therefore it is an element of $\hat{K}$, say a

(iii) We can prove that

$$\lim_{n\to+\infty} a_n = a$$

□

2.3 Complete Fields

Theorem 2.7 (Ostrowski)**.** *Let K be a field which is complete with respect to an archimedean valuation. Then there is an isomorphism σ from K onto $\mathbb{R}$ or C satisfying*

$$|a| = |\sigma(a)|^s \qquad \text{for all } a \in K$$

for some fixed $s \in (0, 1]$.

We will then restrict our attention to the non-archimedean case

Theorem 2.8. *If $\mathcal{O} \subseteq K$,respectively $\hat{\mathcal{O}} \subseteq \hat{K}$,is the valuation ring of ν,respectively of $\hat{\nu}$, and $\mathfrak{p}$, respectively $\hat{\mathfrak{p}}$, is the maximal ideal, then one has*

$$\hat{\mathcal{O}}/\hat{\mathfrak{p}} \simeq \mathcal{O}/\mathfrak{p}$$

and,if ν is discrete, one has furthermore

$$\hat{\mathcal{O}}/\hat{\mathfrak{p}}^n \simeq \mathcal{O}/\mathfrak{p}^n$$

The gist of this section is to show that, in general, many of the features of the p-adics can be easily generalized to the case of non-archimedean complete fields.

Theorem 2.9. *Let $R \subseteq \mathcal{O}$ be a system of representatives for $\kappa = \mathcal{O}/\mathfrak{p}$ such that $0 \in R$, and let $\pi \in \mathcal{O}$ be a prime element . Then every $x \neq 0$ in K admits a unique representation as a convergent series*

$$x = \pi^m(a_0 + a_1\pi + a_2\pi^2 + \ldots)$$

where $a_i \in R$, $a_0 \neq 0$ and $m \in \mathbb{Z}$.

Example. In $\mathbb{Q}_p$ we have $R = 0, \ldots, p-1$ and we have seen that we can write

$$x = p^m(a_0 + a_1 p + a_2 p^2 + \ldots)$$

Example. In the case of the rational function field $k(X)$ and the valuation attached to the prime ideal $\mathfrak{p} = (X - a)$ we may take as a system of representatives the field k itself and the completion turns out to be the ring f formal power series $k((X))$ consisting of Laurent series expansions

$$f(X) = (X-a)^m(a_0 + a_1(X-a) + a_2(X-a)^2 + \ldots)$$

Theorem 2.10. *The canonical mapping $\mathcal{O} \to \varprojlim \mathcal{O}/\mathfrak{p}^n$ is an isomorphism and an homeomorphism.*

The goal of the talk will be to study field extensions of a complete non-archimedean field. This means that we have to turn to the question of factoring algebraic equations.

Let K again be a field which is complete with respect to a non-archimedean valuation ν. Let $\mathcal{O}$ be the corresponding valuation ring with maximal ideal $\mathfrak{p}$ and residue class field $\kappa = \mathcal{O}/\mathfrak{p}$.

We call a polynomial $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathcal{O}[x]$ primitive if $f(x) \not\equiv 0 \bmod \mathfrak{p}$ and

$$|f| = \max\{|a_0|, \ldots, |a_n|\} = 1$$

Theorem 2.11 (Hensel's Lemma)**.** *If a primitive polynomial $f(x) \in \mathcal{O}[x]$ admits modulo $\mathfrak{p}$ a factorization*

$$f(x) \equiv \bar{g}(x)\bar{(}x) \mod \mathfrak{p}$$

into relatively prime polynomials $\bar{g}, \bar{h} \in \kappa[x]$, then $f(x)$ admits a factorization

$$f(x) = g(x)h(x)$$

into polynomials $g, h \in \mathcal{O}[x]$ such that $\deg(g) = \deg(\bar{g})$ and $g(x) \equiv \bar{g}(x)$ mod $\mathfrak{p}$ and $h(x) \equiv \bar{h}(x)$ mod $\mathfrak{p}$.

Corollary 2.12. *Let the field K be complete with respect to the non-archimedean valuation $|\ |$. Then, for every irreducible polynomial $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in K[x]$ such that $a_0 a_n \neq 0$, one has*

$$|f| = \max\{|a_0|, |a_n|\}$$

In particular, $a_n = 1$ and $a_0 \in \mathcal{O}$ imply that $f \in \mathcal{O}[x]$.

Theorem 2.13. *Let K be complete with respect to the valuation $|\ |$. Then $|\ |$ may be extended in a unique way to a valuation of any given algebraic extension L of K. This extension is given by*

$$|a| = \sqrt[n]{\left|\mathrm{N}_{L/K}(a)\right|}$$

when L/K has finite degree n. In this case L is again complete.

Sketch of the proof. Firs of all if $|\ |$ is archimedean then $K = \mathbb{R}$ or $\mathbb{C}$ and the result is known.

Therefore we reduce to the non-archimedean case. Since every algebraic extension L/K is union of its finite sub-extensions we assume that $[L : K] = n$ is finite.

$$\begin{array}{ll} L & \mathcal{O}_L = \text{integral closure of } \mathcal{O}_K \\ \Big| & \Big| \\ K & \mathcal{O}_K = \text{valuation ring of } |\ | \end{array}$$

EXISTENCE It suffices to show that the formula $|a| = \sqrt[n]{\left|\mathrm{N}_{L/K}(a)\right|}$ defines a valuation. Clearly $|a| = 0 \Leftrightarrow a = 0$ and $|ab| = |a|\,|b|$ hold. The strong triangle inequality comes from the fact that $\mathcal{O}_L = \{\alpha \in L \mid \mathrm{N}_{L/K}(\alpha) \in \mathcal{O}_K\}$.

Obviously this valuation has $\mathcal{O}_L$ as valuation ring.

UNIQUENESS Let $|a|'$ another valuation with $\mathcal{O}'_L$ its valuation ring. Let $\mathcal{P}$ and $\mathcal{P}'$ be the maximal ideals of $\mathcal{O}_L$ and $\mathcal{O}'_L$. We show that $\mathcal{O}_L \subseteq \mathcal{O}'_L$.

Let $\alpha \in \mathcal{O}_L \setminus \mathcal{O}'_L$ and let $f(x) = x^d + \ldots + a_1 x + a_0$ be the minimal polynomial of α over K. Then $a_0, \ldots, a_{d-1} \in \mathcal{O}_K$ and $\alpha^{-1} \in \mathcal{P}'$ hence

$$1 = -\left(a_{d-1}\alpha^{-1} + \ldots + a_1(\alpha^{-1})^{d-1} + a_0(\alpha^{-1})^d\right) \in \mathcal{P}'$$

which is a contradiction.

Thus $\mathcal{O}_L \subseteq \mathcal{O}'_L \implies |\alpha| \leq 1$ implies $|\alpha|' \leq 1 \implies$ the two valuation are equivalent but, since they coincide on K, they are equal.

□

3 Analysis in $\mathbb{Q}_p$

In this section we will see some properties of sequences and series in $\mathbb{Q}_p$. Recall from the previous section that $\mathbb{Q}_p$ is a complete metric space. Hence every Cauchy sequence converges and therefore the set of the convergent sequences is the set of the Cauchy sequences. We will also see that we have some much better properties in $\mathbb{Q}_p$, than we are used to in the real case. The first example is the characterization of the Cauchy sequences.

Theorem 3.1. *A sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}_p$ is a Cauchy sequence (i.e. converges), if and only if*

$$\lim_{n\to+\infty} |a_{n+1} - a_n|_p = 0$$

This is obviously not true in $\mathbb{R}$. For example take $a_n = \sum_{k=1}^{n} \frac{1}{n}$ the n-th harmonic number. Then $\lim\limits_{n\to+\infty} |a_{n+1} - a_n| = \lim\limits_{n\to+\infty} \frac{1}{n+1} = 0$ but as we know the $\{a_n\}$ is not Cauchy.

Definition. Let the series $\sum_{n=0}^{+\infty} a_n$ be in $\mathbb{Q}_p$. The sum converges if the sequence of its partial sums converges in $\mathbb{Q}_p$, i.e.,

$$\lim_{n\to+\infty} |S_{N+1} - S_N|_p = 0 \qquad \text{where} \qquad S_N = \sum_{n=0}^{N} a_n$$

The sum converges absolutely if $\sum_{n=0}^{+\infty} |a_n|_p$ converges in $\mathbb{R}$.

As we are used to in $\mathbb{R}$, convergence follows from absolute convergence by the triangle inequality.

Proposition 3.2. *If the series $\sum_{n=0}^{+\infty} |a_n|_p$ converges absolutely (in $\mathbb{R}$), then $\sum_{n=0}^{+\infty} a_n$ converges in $\mathbb{Q}_p$.*

Proposition 3.3. *A series $\sum_{n=0}^{+\infty} a_n$ in $\mathbb{Q}_p$ converges in $\mathbb{Q}_p$ if and only if $\lim\limits_{n\to+\infty} a_n = 0$ and in this case*

$$\left|\sum_{n=0}^{+\infty} a_n\right|_p \leq \max_{n\in\mathbb{N}} |a_n|_p$$

4 Extensions

In this chapter we discuss the question whether, and in how many ways, a valuation ν of a field K can be extended to another field L containing K.

For a complete field K and an algebraic extension L/K we have shown in Section 2 that this extension is unique.

Notation. We denote K the base field with valuation ν. K_ν will be its completion with respect to ν and $\bar{K}_\nu$ will be the algebraic closure of K_ν. The canonical extension of ν to K_ν will be again denoted as ν while its unique extension to $\bar{K}_\nu$ will be $\bar{\nu}$.

Let L/K be an algebraic extension. Choosing a K-embedding $\tau : L \to \bar{K}_\nu$ we obtain by restriction of $\bar{\nu}$ to $\tau(L)$ an extension $\omega = \bar{\nu} \circ \tau$ of the valuation ν to L

The mapping $\tau : L \to K$ is obviously continuous with respect to this valuation. It extends in a unique way to a continuous K-embedding

$$\tau : L_\omega \to \bar{K}_\nu$$

where, in the case of an infinite extension L/K, L_ω does not mean the completion of L with respect to ω, but the union $L = \bigcup_i L_\omega^{(i)}$ of the completions $L_\omega^{(i)}$ of all the finite sub-extensions $L^{(i)}$ of L/K. This union will be called the localization of L with respect to ω.

$$\begin{tikzcd}
 & L_\omega \arrow[dd, dash] \\
L \arrow[ur, dash] \arrow[dd, dash] & \\
 & K_\nu \\
K \arrow[ur, dash] &
\end{tikzcd}$$

The canonical extension of the valuation ω from L to L_ω, is precisely the unique extension of the valuation ν from K_ν to the extension L_ω/K_ν. We have

$$L_\omega = LK_\nu$$

We saw that every K-embedding $\tau : L \to \bar{K}_\nu$ gave us an extension $\omega = \bar{\nu} \circ \tau \circ \nu$. For every automorphism $\sigma \in \mathcal{G}\mathrm{al}(\bar{K}_\nu|K_\nu)$, we obtain with the composite

$$L \xrightarrow{\tau} \bar{K}_\nu \xrightarrow{\sigma} \bar{K}_\nu$$

a new K-embedding τ'.It will be said to be conjugate to τ over $K\nu$.

Theorem 4.1. *Let L/K be an algebraic field extension and ν a valuation of K. Then one has:*

(i) *Every extension ω of the valuation ν arises as the composite $\omega = \bar{\nu} \circ \tau$ for some K-embedding $\tau : L \to K$.*

(ii) *Two extensions $\bar{\nu} \circ \tau$ and $\bar{\nu} \circ \tau'$ are equal if and only if τ and τ' are conjugate over K.*

Let $L = K(\alpha)$ be generated by the zero of an irreducible polynomial $f(x) \in K[x]$ and let

$$f(x) = f_1(x)^{m_1} \cdot \ldots \cdot f_r(x)^{m_r}$$

be the decomposition of $f(x)$ into irreducible factors over the completion $K\nu$ The K-embeddings $\tau : L \to \bar{K}_\nu$ are given by the zeroes β of $f(x)$ which lie in $\bar{K}_\nu$: $\tau(\alpha) = \beta$

Theorem 4.2. *Suppose the extension L/K is generated by α as above Then the valuations $\omega_1, \ldots, \omega_r$ extending ν to L corresponds one-to-one to the irreducible factors $f_1, \ldots, f_r$ in the decomposition of $f(x)$ over the completion K_ν.*

Let L/K be again an arbitrary finite extension. We will write $\omega|\nu$ to indicate that ω is an extension of ν. The inclusions $L \hookrightarrow L_\omega$ induce homomorphisms $L \otimes_K K_\nu \to L_\omega$ via $a \otimes b \to ab$ and hence a canonical homomorphism

$$\varphi : L \otimes_K K_\nu \longrightarrow \prod_{\omega|\nu} L_\omega$$

Proposition 4.3. *If L/K is separable then $L \otimes_K K_\nu \simeq \prod_{\omega|\nu} L_\omega$. Further*

$$[L : K] = \sum_{\omega|\nu} [L_\omega : K_\nu]$$

$$\mathrm{N}_{L/K}(\alpha) = \prod_{\omega|\nu} \mathrm{N}_{L_\omega/K_\nu}(\alpha)$$

$$\mathrm{Tr}_{L/K}(\alpha) = \sum_{\omega|\nu} \mathrm{Tr}_{L_\omega/K_\nu}(\alpha)$$

If ν is a non-archimedean valuation, we define the ramification index of an extension $\omega|\nu$ by

$$e_\omega = \big(\omega(L^\times) : \nu(K^\times)\big)$$

and the inertia degree by

$$f_\omega = [\lambda_\omega : \kappa]$$

where $\lambda_\omega\omega$ is the residue field of ω. One obtain

Proposition 4.4. *It holds*

$$[L : K] = \sum_{\omega|\nu} e_\omega f_\omega$$

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